

A BSDE APPROACH TO FAIR BILATERAL PRICING UNDER ENDOGENOUS COLLATERALIZATION

Tianyang Nie and Marek Rutkowski*
School of Mathematics and Statistics
University of Sydney
Sydney, NSW 2006, Australia

1 December 2014

Abstract

Results from Nie and Rutkowski [12, 14] are extended to the case of the margin account, which may depend on the contract's value for the hedger and/or the counterparty (recall that the collateral was given exogenously in [12, 14]). The present work generalizes also the papers by Bergman [1], Mercurio [11] and Piterbarg [16]. Using the comparison theorems for BSDEs, we derive inequalities for the unilateral prices and we give the range for its fair bilateral prices. We also establish results yielding the link to the market model with a single interest rate. In the case where the collateral amount is negotiated between the counterparties, so that it depends on their respective unilateral values, the backward stochastic viability property studied by Buckdahn et al. [4] is used to derive the bounds on fair bilateral prices.

Keywords: fair bilateral prices, borrowing rate, lending rate, margin agreement, BSDE, BSVP

Mathematics Subjects Classification (2010): 91G20, 91G80

*The research of Tianyang Nie and Marek Rutkowski was supported under Australian Research Council's Discovery Projects funding scheme (DP120100895).

1 Introduction

In Bielecki and Rutkowski [2], the authors introduced a generic nonlinear market model which includes several risky assets, multiple funding accounts, as well as the margin account for collateral (for related studies by other authors, see also [3, 5, 6, 7, 8, 15, 16]). We continue their study by examining the pricing and hedging of a derivative contract from the perspective of the hedger and his counterparty. Since we work within a nonlinear trading framework, the prices computed by the two parties of a contract do not necessarily coincide and thus our goal is to compare these prices and to derive the range for no-arbitrage bilateral prices. As emphasized in [2, 12, 13], the initial endowments of the hedger and the counterparty become important factors in arbitrage pricing in the nonlinear setup. In [12, 14], we studied collateralized contracts in the model with partial netting and Bergman's model, respectively. Using comparison theorems for BSDEs, we derived the range for either fair bilateral prices or bilaterally profitable prices. It should be stressed that in [12, 14], the collateral amount was assumed to be exogenously specified and thus it was independent of unilateral values of the contract for the two parties. By contrast, we study here a more realistic situation where the collateral is endogenous, meaning that it may depend on the marked-to-market value of the contract either for one party (say, the hedger) or it is negotiated between them. Although we focus here on two particular instances of market models, it is clear that the method developed in this work can be applied to a large variety of models and/or collateral covenants.

Motivated by the seminal paper by Bergman [1], we first consider an extension of his trading model to the case of endogenous collateral. To the best of our knowledge, the case of endogenous collateral was not studied in the existing literature, except for the special case of the proportional collateral examined by Piterbarg [16] and Mercurio [11]. We give here essential extensions of their results using the BSDE approach. First, we consider general collateralized contracts, rather than path-independent European claims. Second, in [11], the collateralization of the hedger (resp., the counterparty) was postulated to be a constant proportion of the hedger's (resp., the counterparty's) value, which apparently means that the two parties either post or receive the collateral amounts specified by two different margin accounts. This is clearly inconsistent with the market practice where the collateral amount posted by one party coincides with the amount received by another party. We derive inequalities satisfied by unilateral prices of a contract and we give the range for its fair bilateral prices. We show that if the collateralization depends on the values for the hedger and the counterparty, the backward stochastic viability property (BSVP) plays an important role in derivation of pricing inequalities. Motivated by results from the papers by Buckdahn et al. [4] and Hu and Peng [10], we obtain the range of fair bilateral prices for European contingent claims. In the second step, we consider the market model with partial netting under the assumption of full rehypothecation of the cash collateral. Once again, in contrast with our previous work [12], we study here the case of the collateral depending on the hedger's value and/or counterparty's value. We establish similar results as for Bergman's model. It is worth noting, however, that the model with partial netting enjoys some additional properties with respect to the class of monotone contracts, which are not necessarily shared by Bergman's model. This emphasizes the impact of asset-specific funding costs on properties of hedging strategies and prices of contracts.

The work is organized as follows. In Section 2, we recall some definitions and assumptions introduced in our previous works [2, 12, 14]. In Section 3 and 4, we examine extensions of the model studied by Bergman [1] and Mercurio [11]. In Section 3, we consider the case where the collateral depends only on the hedger's value and we establish inequalities for unilateral prices of a general contract. Moreover, we extend the results from [11] regarding the relationship to the market model with a single uncertain interest rate. In Section 4, we study the case where the collateral depends on both the hedger's and the counterparty's values under the assumption that the risky asset is driven by a Brownian motion. Using the BSVP technique from [4], we derive the inequalities for unilateral prices. In Sections 5 and 6, we examine the model with partial netting and we obtain similar results for the range of fair bilateral prices. We also show that the model with partial netting has some additional properties of independence of the initial endowment and/or positive homogeneity with respect to particular classes of contracts. We thus conclude that no unified theory can be developed in the non-linear framework, so that each particular setup should be studied on a standalone basis.

2 Preliminaries

We provide here a very brief summary of concepts and notation introduced in [2, 12, 14]. For more details and explanations, the reader is referred to the original papers. Let $T > 0$ be a fixed finite trading horizon date for our model of the financial market. We denote by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$ models the flow of information available to all traders. For convenience, we assume that the initial σ -field \mathcal{G}_0 is trivial. Moreover, all processes introduced in what follows are implicitly assumed to be \mathbb{G} -adapted and any semimartingale is assumed to be càdlàg. As in [12, 14], for any $i = 1, 2, \dots, d$, we use the following notation for the market data:

A – a *bilateral financial contract*, or simply a *contract*. The process A is finite variation and it represents the *cumulative cash flows* of a given contract from time 0 till its maturity date T ,

C – the cash collateral, specified as a \mathbb{G} -adapted process satisfying $C_T = 0$,

S^i – the *ex-dividend price* of the i th risky asset with the *cumulative dividend stream* A^i ,

B^l (resp., B^b) – the *lending* (resp., *borrowing*) *cash account*,

$B^{i,l}$ (resp., $B^{i,b}$) – the *lending* (resp., *borrowing*) *funding account* associated with the i th risky asset,

B^c – the process specifying the interest paid/received on the collateral account received/posted.

Assumption 2.1 We work throughout under the following assumptions:

- (i) S^i is a semimartingale and A^i is a process of finite variation with $A_0^i = 0$.
- (ii) the processes $B^l, B^b, B^{i,l}, B^{i,b}$ and B^c are strictly positive, continuous processes of finite variation with $B_0^l = B_0^b = B_0^{i,l} = B_0^{i,b} = B_0^c = 1$.
- (iii) in the case of a model with partial netting, we also assume that $B^{i,l} = B^l$ for every i ,
- (iv) $dB_t^l = r_t^l B_t^l dt$, $dB_t^b = r_t^b B_t^b dt$, $dB_t^{i,b} = r_t^{i,b} B_t^{i,b} dt$ and $dB_t^c = r_t^c B_t^c dt$, for some \mathbb{G} -adapted and bounded processes $r^l, r^b, r^{i,b}$ and r^c . Moreover, we postulate that $0 \leq r^l \leq r^b$ and $r^l \leq r^{i,b}$.

We define the interest process of the margin account by setting $F_t^C := -\int_0^t r_u^c C_u du$ for every t and we denote $A^C := A + C + F^C$. For a *collateralized hedger's trading strategy* (x, φ, A, C) , we write:

- (i) $V_t(x, \varphi, A, C)$ – the hedger's wealth at time t ,
- (ii) $V_t^p(x, \varphi, A, C)$ – the value of hedger's portfolio at time t .

Note that $V_t^p(x, \varphi, A, C) - V_t(x, \varphi, A, C) = C_t$ measures the impact of the margin account represented by the collateral amount C_t on the hedger's trading strategy under the standing assumption of *full rehypothecation*. Finally, we set $V_t^0(x) := xB_T^l \mathbf{1}_{\{x \geq 0\}} + xB_T^b \mathbf{1}_{\{x < 0\}}$ where $x = x_1$ (resp., $x = x_2$) is the *initial endowment* of the hedger (resp., the counterparty) at time 0.

Definition 2.1 Any \mathcal{G}_t -measurable random variable for which a replicating strategy for A over $[t, T]$ exists is called the *hedger's ex-dividend price* at time t for a contract (A, C) and it is denoted by $P_t^h(x_1, A, C)$, so that for some self-financing trading strategy φ , which replicates (A, C) , we have

$$V_T(V_t^0(x_1) + P_t^h(x_1, A, C), \varphi, A - A_t, C) = V_T^0(x_1).$$

For an arbitrary level x_2 of the counterparty's initial endowment and a strategy $\tilde{\varphi}$ replicating $(-A, -C)$, the *counterparty's ex-dividend price* $P_t^c(x_2, -A, -C)$ at time t for a contract $(-A, -C)$ is implicitly given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \tilde{\varphi}, -A + A_t, -C) = V_T^0(x_2).$$

By a *fair bilateral price*, we mean any level of the price at which no arbitrage opportunity arises for either of the two parties. Hence the range of fair bilateral prices at time t is formally defined as follows.

Definition 2.2 The \mathcal{G}_t -measurable interval

$$\mathcal{R}_t^f(x_1, x_2) := [P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)]$$

is called the *range of fair bilateral prices* at time t of an OTC contract (A, C) between the hedger and the counterparty.

3 Bergman's Model with Hedger's Collateral

In Sections 3 and 4, we consider an extended version of the model studied by Bergman [1]. For a detailed analysis of this model, we refer to the recent work by Nie and Rutkowski [14]. Note that in this framework the funding accounts $B^{i,l}$ and $B^{i,b}$ are not introduced.

Following [2, 12, 14], we introduce the auxiliary processes $\tilde{S}^{i,l,\text{cld}}$ and $\tilde{S}^{i,b,\text{cld}}$, which are given by the following expressions, for $i = 1, 2, \dots, d$,

$$\tilde{S}_t^{i,l,\text{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1} dA_u^i$$

and

$$\tilde{S}_t^{i,b,\text{cld}} := (B_t^b)^{-1} S_t^i + \int_{(0,t]} (B_u^b)^{-1} dA_u^i.$$

It is easy to see that the dynamics of these processes are

$$d\tilde{S}_t^{i,l,\text{cld}} = (B_t^l)^{-1} (dS_t^i + dA_t^i - r_t^l S_t^i dt) \quad (3.1)$$

and

$$d\tilde{S}_t^{i,b,\text{cld}} = (B_t^b)^{-1} (dS_t^i + dA_t^i - r_t^b S_t^i dt). \quad (3.2)$$

As in [14], we consider an arbitrary self-financing trading strategy $\varphi = (\xi^1, \dots, \xi^d, \varphi^l, \varphi^b, \eta^l, \eta^b)$ where $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ and $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$. Since we assume here that $B_t^{c,l} = B_t^{c,b} = B_t^c$, a trading strategy φ reduces to $(\xi^1, \dots, \xi^d, \varphi^l, \varphi^b, \eta)$ where $\eta_t = -(B_t^c)^{-1} C_t$. Let us denote

$$A_t^{C,l} := \int_{(0,t]} (B_u^l)^{-1} dA_u^C, \quad A_t^{C,b} := \int_{(0,t]} (B_u^b)^{-1} dA_u^C.$$

From Proposition 2.9 in [14], it is known that the process $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$ satisfies

$$dY_t^l = \sum_{i=1}^d Z_t^{l,i} d\tilde{S}_t^{i,l,\text{cld}} + G_l(t, Y_t^l, Z_t^l) dt + dA_t^{C,l} \quad (3.3)$$

where $Z^{l,i} := \xi^i$, $i = 1, 2, \dots, d$ and the generator G_l is given by the following expression, for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$G_l(t, y, z) = \sum_{i=1}^d r_t^l (B_t^l)^{-1} z^i S_t^i + (B_t^l)^{-1} \left(r_t^l \left(y B_t^l - \sum_{i=1}^d z^i S_t^i \right)^+ - r_t^b \left(y B_t^l - \sum_{i=1}^d z^i S_t^i \right)^- \right) - r_t^l y.$$

Similarly, the process $Y^b := (B^b)^{-1} V^p(x, \varphi, A, C)$ is governed by

$$dY_t^b = \sum_{i=1}^d Z_t^{b,i} d\tilde{S}_t^{i,b,\text{cld}} + G_b(t, Y_t^b, Z_t^b) dt + dA_t^{C,b} \quad (3.4)$$

where $Z^{b,i} := \xi^i$, $i = 1, 2, \dots, d$ and the generator G_b equals, for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$G_b(t, y, z) = \sum_{i=1}^d r_t^b (B_t^b)^{-1} z^i S_t^i + (B_t^b)^{-1} \left(r_t^l \left(y B_t^b - \sum_{i=1}^d z^i S_t^i \right)^+ - r_t^b \left(y B_t^b - \sum_{i=1}^d z^i S_t^i \right)^- \right) - r_t^b y.$$

Recall that the initial endowment of the hedger (resp., the counterparty) is denoted by x_1 (resp., x_2). Without loss of generality, we assume throughout that $x_1 \geq 0$ and we consider an arbitrary level of x_2 . Furthermore, in Sections 3 and 5, we work under the following standing assumption of *hedger's collateral*, that is, the situation where the collateral amount only depends on the hedger's wealth $V^h := V(x_1, \varphi, A, C)$.

Assumption 3.1 The *hedger's collateral* C is given by the equality

$$C_t = q(V_t^0(x_1) - V_t^h) \quad (3.5)$$

for some uniformly Lipschitz continuous function $q : \mathbb{R} \rightarrow \mathbb{R}$ such that $q(0) = 0$.

Example 3.1 For instance, the hedger's collateral C can be specified as in [2] (see equation (4.10) therein) through the following expression

$$C_t = (1 + \alpha_1)(V_t^0(x) - V_t^h)^+ - (1 + \alpha_2)(V_t^0(x) - V_t^h)^-$$

for some bounded *haircut* processes α_1, α_2 , so that $q(y) = (1 + \alpha_1)y^+ - (1 + \alpha_2)y^-$. It is clear that q is uniformly Lipschitz continuous and $q(0) = 0$. The case of the fully collateralized contract, from the perspective of the hedger, is obtained by taking $q(y) = y$, that is, by setting $\alpha_1 = \alpha_2 = 0$.

3.1 Initial Endowments of Equal Signs

We first examine the case where the initial endowments are of the same sign, specifically, we assume that $x_1 \geq 0$ and $x_2 \geq 0$. The next assumption postulates the existence of a ‘martingale measure’ in the present setup. All probability measures are assumed to be defined on (Ω, \mathcal{G}_T) .

Assumption 3.2 There exists a probability measure $\tilde{\mathbb{P}}^l$ equivalent to \mathbb{P} such that the processes $\tilde{S}^{i,l,\text{cld}}, i = 1, 2, \dots, d$ are $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -local martingales.

The following result is borrowed from [14] (see Proposition 2.1 therein). Let us stress that the arbitrage-free property is here understood in the sense of [2] (see Section 3.1 therein).

Proposition 3.1 *If the initial endowments satisfy $x_1 \geq 0, x_2 \geq 0$ and Assumption 3.2 is valid, then Bergman's model is arbitrage-free for the hedger and the counterparty with respect to any contract (A, C) .*

In order to address the issue of bilateral pricing using a BSDE approach, we need to impose additional assumptions on the dynamics of risky assets. We will work under the following assumption regarding the quadratic variation process for continuous martingales $\tilde{S}^{l,\text{cld}}$. Note that $*$ stands for the transposition and, as in [12, 14], we define the matrix-valued process \mathbb{S} given by

$$\mathbb{S}_t := \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & S_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_t^d \end{pmatrix}.$$

Assumption 3.3 We postulate that:

- (i) there exists a probability measure $\tilde{\mathbb{P}}^l$ equivalent to \mathbb{P} such that $\tilde{S}^{l,\text{cld}}$ is a continuous, square-integrable, $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -martingale and has the predictable representation property with respect to the filtration \mathbb{G} under $\tilde{\mathbb{P}}^l$,
- (ii) there exists an $\mathbb{R}^{d \times d}$ -valued, \mathbb{G} -adapted process m^l such that

$$\langle \tilde{S}^{l,\text{cld}} \rangle_t = \int_0^t m_u^l (m_u^l)^* du \quad (3.6)$$

where the process $m^l(m^l)^*$ is invertible and satisfies $m^l(m^l)^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$. Here σ is a d -dimensional square matrix of \mathbb{G} -adapted processes satisfying the *ellipticity condition*: there exists a constant $\Lambda > 0$

$$\sum_{i,j=1}^d (\sigma_t \sigma_t^*)_{ij} a_i a_j \geq \Lambda |a|^2 = \Lambda a^* a, \quad \forall a \in \mathbb{R}^d, t \in [0, T]. \quad (3.7)$$

Following Nie and Rutkowski [13], but with $Q_t = t$, we denote by $\widehat{\mathcal{H}}_0^{2,d}$ the subspace of all \mathbb{R}^d -valued, \mathbb{G} -adapted processes X such that

$$|X|_{\widehat{\mathcal{H}}_0^{2,d}}^2 := \mathbb{E}_{\mathbb{P}} \left[\int_0^T |X_t|^2 dt \right] < \infty \quad (3.8)$$

and, for brevity, we write $\widehat{\mathcal{H}}_0^2 := \widehat{\mathcal{H}}_0^{2,1}$. Also, let \widehat{L}_0^2 stand for the space of all real-valued, \mathcal{G}_T -measurable random variables η such that $|\eta|_{\widehat{L}_0^2}^2 = \mathbb{E}_{\mathbb{P}}(\eta^2) < \infty$.

Definition 3.1 For any probability measure \mathbb{Q} , we denote by $\mathcal{A}(\mathbb{Q})$ the following class of a real-valued, \mathbb{G} -adapted processes $\mathcal{A}(\mathbb{Q}) := \{X \in \widehat{\mathcal{H}}_0^2 \text{ and } X_T \in \widehat{L}_0^2 \text{ under } \mathbb{Q}\}$.

Definition 3.1 will serve to define the class of *admissible* contracts with the choice of \mathbb{Q} depending on a particular setup at hand. Let us stress that for any contract (A, C) the statement that $A \in \mathcal{A}(\mathbb{Q})$ will mean that the process $A - A_0$ of future cash flows belongs to the class $\mathcal{A}(\mathbb{Q})$. Recall that the initial cash flow A_0 of a contract (A, C) represents its initial price, so that is not given a priori.

For the reader's convenience, we first recall a result concerning the case of an exogenous collateral C (see Propositions 3.1 and 3.2 in [14], as well as Proposition 5.2 in [2]).

Proposition 3.2 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1 and 3.3 be valid. Then for any contract (A, C) such that $A^{C,l} \in \mathcal{A}(\widetilde{\mathbb{P}}^l)$, the hedger's ex-dividend price equals $P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$ where the pair $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to the BSDE*

$$\begin{cases} dY_t^{h,l,x_1} = Z_t^{h,l,x_1,*} d\widetilde{S}_t^{l,cld} + G_l(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt + dA_t^{C,l}, \\ Y_T^{h,l,x_1} = x_1, \end{cases} \quad (3.9)$$

and the counterparty's ex-dividend price equals $P^c(x_2, -A, -C) = -B^l(Y^{c,l,x_2} - x_2) + C$ where the pair $(Y^{c,l,x_2}, Z^{c,l,x_2})$ is the unique solutions to the BSDE

$$\begin{cases} dY_t^{c,l,x_2} = Z_t^{c,l,x_2,*} d\widetilde{S}_t^{l,cld} + G_l(t, Y_t^{c,l,x_2}, Z_t^{c,l,x_2}) dt - dA_t^{C,l}, \\ Y_T^{c,l,x_2} = x_2. \end{cases} \quad (3.10)$$

In the next result, the hedger's collateral C is given by equation (3.5). Note that the generator g_l depends explicitly on the process Y^1 , which in turn is defined as a part of the solution of BSDE (3.11). This means that the counterparty's BSDE (3.12) is coupled with the hedger's BSDE (3.11). It is thus crucial to note that the hedger's price $P^h(x_1, A, C)$ depends only on his initial endowment x_1 . By contrast, the counterparty's price depends on both initial endowments, x_1 and x_2 , so that it would be suitable to denote it as $P^c(x_1, x_2, -A, -C)$. However, for ease of notation, we shall write $P^c(x_2, -A, -C)$, while keeping in mind that this process depends on x_1 as well.

Proposition 3.3 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1 and 3.3 be valid. Then for any contract (A, C) such that $A \in \mathcal{A}(\mathbb{P}^l)$, the hedger's ex-dividend price equals $P^h := P^h(x_1, A, C) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE*

$$\begin{cases} dY_t^1 = Z_t^{1,*} d\widetilde{S}_t^{l,cld} + f_l(t, x_1, Y_t^1, Z_t^1) dt + dA_t, \\ Y_T^1 = 0, \end{cases} \quad (3.11)$$

with the generator f_l given by

$$\begin{aligned} f_l(t, x_1, y, z) = & r_t^l (B_t^l)^{-1} z^* S_t - x_1 B_t^l r_t^l - r_t^c q(-y) \\ & + r_t^l \left(y + q(-y) + x_1 B_t^l - (B_t^l)^{-1} z^* S_t \right)^+ - r_t^b \left(y + q(-y) + x_1 B_t^l - (B_t^l)^{-1} z^* S_t \right)^- \end{aligned}$$

and the counterparty's ex-dividend price equals $P^c := P^c(x_2, -A, -C) = Y^2$ where (Y^2, Z^2) is the unique solution to the BSDE

$$\begin{cases} dY_t^2 = Z_t^{2,*} d\tilde{S}_t^{l,cld} + g_l(t, x_2, Y_t^2, Z_t^2) dt + dA_t, \\ Y_T^2 = 0, \end{cases} \quad (3.12)$$

with the generator g_l given by

$$\begin{aligned} g_l(t, x_2, y, z) &= r_t^l (B_t^l)^{-1} z^* S_t + x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) \\ &\quad - r_t^l \left(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} z^* S_t \right)^+ + r_t^b \left(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} z^* S_t \right)^-. \end{aligned}$$

Proof. Since the collateral amount is not exogenously specified in the present framework, the process C may depend on the hedger's value, and thus Proposition 3.2 does not cover the current situation. However, from the proof of Proposition 5.2 in [2], one can deduce that if BSDEs (3.9) and (3.10) have a unique solution, then the relationships $P^h(x, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$ and $P^c(x_2, -A, -C) = -B^l(Y^{c,l,x_2} - x_2) + C$ are still valid.

It is also worth stressing that we cannot apply directly the results of [13] to solve BSDEs (3.9) and (3.10), since the process A^C depends also on $Y^{h,l,x}$. However, since $P^h := B^l(Y^{h,l,x_1} - x_1) - C$, we have that $P_T^h = 0$ and thus, by letting $\tilde{Z}^{h,l,x_1} := B^l Z^{h,l,x_1}$, we obtain

$$\begin{aligned} dP_t^h &= B_t^l Z_t^{h,l,x_1,*} d\tilde{S}_t^{l,cld} + B_t^l G_l(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}) dt + r_t^l B_t^l (Y_t^{h,l,x_1} - x) dt + dA_t^C - dC_t \\ &= B_t^l Z_t^{h,l,x_1,*} d\tilde{S}_t^{l,cld} + r_t^l Z_t^{h,l,x_1,*} S_t dt + r_t^l \left(Y_t^{h,l,x_1} B_t^l - Z_t^{h,l,x_1,*} S_t \right)^+ dt \\ &\quad - r_t^b \left(Y_t^{h,l,x_1} B_t^l - Z_t^{h,l,x_1,*} S_t \right)^- dt - r_t^l B_t^l Y_t^{h,l,x_1} dt + r_t^l B_t^l (Y_t^{h,l,x_1} - x_1) dt + dA_t + dF_t^C \\ &= \tilde{Z}_t^{h,l,x_1,*} d\tilde{S}_t^{l,cld} + r_t^l \left(P_t^h + C_t + x_1 B_t^l - (B_t^l)^{-1} \tilde{Z}_t^{h,l,x_1,*} S_t \right)^+ dt \\ &\quad - r_t^b \left(P_t^h + C_t + x_1 B_t^l - (B_t^l)^{-1} \tilde{Z}_t^{h,l,x_1,*} S_t \right)^- dt \\ &\quad - x_1 r_t^l B_t^l dt + r_t^l (B_t^l)^{-1} \tilde{Z}_t^{h,l,x_1,*} S_t dt + dA_t - r_t^c C_t dt. \end{aligned}$$

Moreover, by comparing Proposition 5.2 in [2] with equation (3.3), we deduce easily that $Y^{h,l,x_1} = (B^l)^{-1} V^p(x_1, \varphi, A, C)$ and thus

$$P^h = B^l(Y^{h,l,x_1} - x_1) - C = V^p(x_1, \varphi, A, C) - x_1 B^l - C = V(x_1, \varphi, A, C) - x_1 B^l.$$

By applying similar arguments to the counterparty's pricing problem, we obtain the equality $Y^{c,l,x_2} = (B^l)^{-1} V^p(x_2, \tilde{\varphi}, -A, -C)$, which in turn yields

$$P^c(x_2, -A, -C) = -B^l(Y^{c,l,x_2} - x_2) + C = -V(x_2, \tilde{\varphi}, -A, -C) + x_2 B^l.$$

We conclude that if C is given by equation (3.5), then we have $C_t = q(V_t^0(x_1) - V_t^h) = q(-P_t^h)$ and thus the pair $(P^h, \tilde{Z}^{h,l,x_1})$ is a solution to BSDE (3.11). Similarly, for $\tilde{Z}^{c,l,x_2} := -B_t^l Z^{c,l,x_2}$, we deduce that the pair $(P^c, \tilde{Z}^{c,l,x_2})$ satisfies BSDE (3.12).

It remains to verify that BSDEs (3.11) and (3.12) are indeed well-posed. One can check that $f_l(t, x_1, 0, 0) = 0$ and the mapping f_l is uniformly m -Lipschitz generator (for the definition of the uniformly m -Lipschitz generator, see [13]). Consequently, if $A \in \mathcal{A}(\mathbb{P}^l)$ then, using Theorem 3.2 in [13], we conclude that BSDE (3.11) has a unique solution (Y^1, Z^1) such that $(Y^1, m^* Z^1) \in \hat{\mathcal{H}}_0^2 \times \hat{\mathcal{H}}_0^{2,d}$. Similarly, we note that

$$g_l(t, x_2, 0, 0) = x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) - r_t^l \left(-q(-Y_t^1) + x_2 B_t^l \right)^+ + r_t^b \left(-q(-Y_t^1) + x_2 B_t^l \right)^-$$

where q is a uniformly Lipschitz continuous function and $Y^1 \in \hat{\mathcal{H}}_0^2$, so that $g_l(t, x_2, 0, 0) \in \hat{\mathcal{H}}_0^2$. Moreover, the mapping g_l is also a uniformly m -Lipschitz generator, and thus BSDE (3.12) has also a unique solution (Y^2, Z^2) such that $(Y^2, m^* Z^2) \in \hat{\mathcal{H}}_0^2 \times \hat{\mathcal{H}}_0^{2,d}$. \square

We are now in a position to examine the range of fair bilateral prices at time t (see Definition 2.2). It appears that, under mild assumptions, this range is non-empty when the initial endowments of the two parties have the same sign. Let us note that this range may be empty, in general, if the initial endowments are of opposite signs, that is, when $x_1 > 0$ and $x_2 < 0$ (see Proposition 3.9(ii) in Section 3.2).

Proposition 3.4 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1 and 3.3 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^l)$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.}, \quad (3.13)$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty, $\tilde{\mathbb{P}}^l - \text{a.s.}$

Proof. In view of Proposition 3.3 and a suitable version of the comparison theorem for BSDEs (see Theorem 3.3 in [13]), to establish the inequality $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$, $\tilde{\mathbb{P}}^l - \text{a.s.}$, it suffices to show that $g_l(t, x_2, Y^1, Z^1) \geq f_l(t, x_1, Y^1, Z^1)$, $\tilde{\mathbb{P}}^l \otimes \ell - \text{a.e.}$. To demonstrate the latter inequality, we denote

$$\delta := g_l(t, x_2, Y^1, Z^1) - f_l(t, x_1, Y^1, Z^1) = r_t^l B_t^l(x_1 + x_2) - r_t^l(\delta_1^+ + \delta_2^+) + r_t^b(\delta_1^- + \delta_2^-)$$

where

$$\delta_1 := -Y_t^1 - q(-Y_t^1) + B_t^l x_2 + (B_t^l)^{-1} Z_t^{1,*} S_t, \quad \delta_2 := Y_t^1 + q(-Y_t^1) + B_t^l x_1 - (B_t^l)^{-1} Z_t^{1,*} S_t.$$

Since, by Assumption 2.1, the inequality $r^l \leq r^b$ holds, we obtain

$$\delta \geq r_t^l B_t^l(x_1 + x_2) - r_t^l(\delta_1 + \delta_2) = 0,$$

which is the required condition. \square

Remark 3.1 It is clear that analogous results can be established when the collateral depends only on the counterparty's value $V^c := V(x_2, \tilde{\varphi}, -A, -C)$, specifically, when Assumption 3.1 is replaced by the postulate that $C_t = q(V_t^c - V_t^0(x_2))$ for some uniformly Lipschitz continuous function q such that $q(0) = 0$.

Remark 3.2 One can also prove similar results when the initial endowments satisfy $x_1 \leq 0$ and $x_2 \leq 0$, so that they are still of the same sign. The case where the initial endowments have opposite signs is more challenging and it is analyzed in Section 3.2.

3.1.1 European Claims in a Diffusion Model

The pricing and hedging of collateralized European claims in a diffusion model was recently studied by Mercurio [11] (see also Piterbarg [16]). It should be pointed out that the hedger's and counterparty's initial endowments were implicitly assumed to be null in [11]. More importantly, the collateral amount for the hedger (resp., for the counterparty) was specified as a constant proportion of the hedger's (resp., the counterparty's) value, that is, it was postulated in [11] that $C^h = \alpha V^h$ and $C^c = \alpha V^c$ for some $\alpha \in [0, 1]$. Such a specification of the margin account apparently corresponds to the situation where the hedger and the counterparty post/receive collateral of possibly different amounts to/from the third party independently of each other. Obviously, this is inconsistent with the real-life situation where the margin account is common for both parties, so that the collateral amount posted (resp., received) by one party is received (resp., posted) by another party.

For simplicity, let us assume $d = 1$, so that there is only one risky asset, $S = S^1$. This restriction can be relaxed and thus Corollary 3.1 can be easily extended to the multi-asset framework.

Assumption 3.4 We assume that:

(i) the risky asset S has the ex-dividend price dynamics under \mathbb{P} given by the following expression, for $t \in [0, T]$,

$$dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dW_t, \quad S_0 = s \in \mathcal{O}, \quad (3.14)$$

where W is a one-dimensional Brownian motion and \mathcal{O} is the domain of real values that are attainable by the diffusion process S (usually $\mathcal{O} = \mathbb{R}_+$),

(ii) the filtration \mathbb{G} is generated by the Brownian motion W ,

(iii) the coefficients μ and σ are such that SDE (3.14) has a unique strong solution,

(iv) the dividend process equals $A_t^1 = \int_0^t \kappa(u, S_u) du$.

We observe that

$$d\tilde{S}_t^{l, \text{cld}} = (B_t^l)^{-1} (dS_t - r_t^l S_t dt + dA_t^1) = (\mu(t, S_t) + \kappa(t, S_t) - r_t^l S_t) dt + \sigma(t, S_t) dW_t.$$

We denote

$$a_t := (\sigma(t, S_t))^{-1} (\mu(t, S_t) + \kappa(t, S_t) - r_t^l S_t) \quad (3.15)$$

and we suppose that a satisfies Novikov's condition

$$\mathbb{E}_{\mathbb{P}} \left(\exp \left\{ \frac{1}{2} \int_0^T |a_t|^2 dt \right\} \right) < \infty. \quad (3.16)$$

Let us define the probability measure $\tilde{\mathbb{P}}^l$ by setting

$$\frac{d\tilde{\mathbb{P}}^l}{d\mathbb{P}} = \exp \left\{ - \int_0^T a_t dW_t - \frac{1}{2} \int_0^T |a_t|^2 dt \right\}. \quad (3.17)$$

From the Girsanov theorem, the probability measure $\tilde{\mathbb{P}}^l$ is equivalent to \mathbb{P} and the process \tilde{W}^l is the Brownian motion under $\tilde{\mathbb{P}}^l$, where $d\tilde{W}_t^l := dW_t + a_t dt$. It is easy to see that the dynamics of the process $\tilde{S}^{l, \text{cld}}$ under $\tilde{\mathbb{P}}^l$ are

$$d\tilde{S}_t^{l, \text{cld}} = \sigma(t, S_t) d\tilde{W}_t^l$$

and thus $\tilde{S}^{l, \text{cld}}$ is a $(\tilde{\mathbb{P}}^l, \mathbb{G})$ -(local) martingale with the quadratic variation $\langle \tilde{S}^{l, \text{cld}} \rangle_t = \int_0^t |\sigma(u, S_u)|^2 du$. Therefore, if $(\sigma(\cdot, S))^{-1} S$ is bounded, Assumption 3.3 holds, since the Brownian motion \tilde{W}^l is known to have the predictable representation property under $(\mathbb{G}, \tilde{\mathbb{P}}^l)$.

Let us comment on the valuation and hedging of a European contingent claim with the hedger's collateral given by (3.5). A generic European claim pays at its expiration date T the amount H_T to the hedger, so that

$$A_t - A_0 = H_T \mathbf{1}_{[T, T]}(t). \quad (3.18)$$

We find it convenient to denote such a contract as (H_T, C) . From Proposition 3.4, we deduce the following corollary.

Corollary 3.1 *Consider a collateralized European claim (H_T, C) where the random variable H_T is square-integrable under $\tilde{\mathbb{P}}^l$. If $x_1 \geq 0$, $x_2 \geq 0$ and Assumption 3.1 is valid, then the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ for (H_T, C) is non-empty, $\tilde{\mathbb{P}}^l$ -a.s.*

3.1.2 Model with an Uncertain Money Market Rate

We continue working under the assumption that $x_1 \geq 0$ and $x_2 \geq 0$. Let us select any \mathbb{G} -adapted interest rate process satisfying the following condition

$$r_t \in [r_t^l, r_t^b] \quad \text{for every } t \in [0, T]. \quad (3.19)$$

We preserve all other assumptions regarding the market model at hand, including the set of traded risky assets but, for the sake of comparison, we now also consider an additional market model with the single uncertain money market rate r . To be more specific, part (iv) in Assumption 2.1 becomes: $dB_t^l = r_t B_t^l dt$, $dB_t^b = r_t B_t^b dt$ and $dB_t^c = r_t^c B_t^c dt$ for some \mathbb{G} -adapted and bounded processes r , r^l and r^c . Under these assumptions, the hedger and the counterparty have the same ex-dividend price P^r , which does not depend on their initial endowments. Intuitively, this is due to the fact that the situation is now fully symmetric since we deal now with a single interest rate. Formally, the ex-dividend price process $P^r = Y$ now coincides with the unique solution to the BSDE

$$\begin{cases} dY_t = Z_t^* d\tilde{S}_t^{l,\text{clid}} + f(t, Y_t, Z_t) dt + dA_t, \\ Y_T = 0, \end{cases} \quad (3.20)$$

where the generator f is given by the following expression

$$f(t, y, z) = (r_t^l - r_t)(B_t^l)^{-1} z^* S_t - r_t^c q(-y) + r_t(y + q(-y)).$$

The next result is not only more general but, in our opinion, it is also more natural than Proposition 4.1 in Mercurio [11] where, as was already mentioned at the beginning of Section 3.1.1, the collateral amount for each party was tied to his unilateral value of the contract. Note that the prices $P^h(0, A, C)$ and $P^c(0, -A, -C)$ are computed in Bergman's model with differential borrowing and lending rates r^l and r^b under the assumption that $x_1 = x_2 = 0$ and the collateral C is given by (3.5).

Proposition 3.5 *Consider the market model with the money market rate r .*

- (i) *The price P^r of any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^l)$ satisfies $P^r \leq P^h(0, A, C)$, $\tilde{\mathbb{P}}^l$ - a.s.*
- (ii) *If $x_1 = x_2 = 0$ and the function q in equation (3.5) satisfies $(r_t - r_t^c)(q(y_1) - q(y_2)) \leq 0$ for all $y_1 \geq y_2$, then also $P^c(0, -A, -C) \leq P^r$, $\tilde{\mathbb{P}}^l$ - a.s.*

Proof. (i) In view of the comparison theorem for BSDEs (see Theorem 3.3 in [13]), it is sufficient to show that the inequality $f_l(t, 0, Y_t, Z_t) \leq f(t, Y_t, Z_t) \leq g_l(t, 0, Y_t, Z_t)$ holds $\tilde{\mathbb{P}}^l \otimes \ell$ - a.e.. Let us denote

$$\delta_1 := Y_t + q(-Y_t) + B_t^l x_1 - (B_t^l)^{-1} Z_t^* S_t.$$

From the assumption that $r_t \in [r_t^l, r_t^b]$ for all $t \in [0, T]$, we obtain

$$\begin{aligned} \delta &:= f_l(t, x_1, Y_t, Z_t) - f(t, Y_t, Z_t) = r_t(B_t^l)^{-1} Z_t^* S_t - x_1 B_t^l r_t^l + r_t^l \delta_1^+ - r_t^b \delta_1^- - r_t Y_t - r_t q(-Y_t) \\ &\leq r_t(B_t^l)^{-1} Z_t^* S_t - x_1 B_t^l r_t^l + r_t \delta_1 - r_t(Y_t + q(-Y_t)) = (r_t - r_t^l) x_1 B_t^l. \end{aligned}$$

Therefore, if $x_1 = 0$, then $\delta \leq 0$. Consequently, $Y \leq Y^1$ and thus $P^r \leq P^h(0, A, C)$.

(ii) We now assume that the hedger's initial endowment is null and we examine the pricing problem for the counterparty. Recall that we postulate that $C_t = q(-Y^1) = q(-P^h(0, A, C))$. Let us denote

$$\delta_2 := -Y_t - q(-Y_t^1) + B_t^l x_2 + (B_t^l)^{-1} Z_t^* S_t.$$

From $r_t \in [r_t^l, r_t^b]$, we obtain

$$\begin{aligned} \tilde{\delta} &:= f(t, Y_t, Z_t) - g_l(t, x_2, Y_t, Z_t) \\ &= -r_t(B_t^l)^{-1} Z_t^* S_t - r_t^c q(-Y_t) + r_t^c q(-Y_t^1) + r_t(Y_t + q(-Y_t)) - x_2 B_t^l r_t^l + r_t^l \delta_2^+ - r_t^b \delta_2^- \\ &\leq -r_t(B_t^l)^{-1} Z_t^* S_t - r_t^c q(-Y_t) + r_t^c q(-Y_t^1) + r_t(Y_t + q(-Y_t)) - x_2 B_t^l r_t^l + r_t \delta_2 \\ &= (r_t - r_t^l) x_2 B_t^l + (r_t - r_t^c)(q(-Y_t) - q(-Y_t^1)). \end{aligned}$$

Since $(r_t - r_t^c)(q(y_1) - q(y_2)) \leq 0$ for all $y_1 \geq y_2$ and $Y \leq Y^1$, we have $(r_t - r_t^c)(q(-Y_t) - q(-Y_t^1)) \leq 0$. Therefore, if $x_2 = 0$, then $\tilde{\delta} \leq 0$. We conclude that $P^c(0, -A, -C) \leq P^r$. \square

Remark 3.3 Note that the condition $(r_t - r_t^c)(q(y_1) - q(y_2)) \leq 0$ for all $y_1 \geq y_2$ can be easily ensured. For instance, if q is an increasing function (e.g., the one given in Example 3.1 when $\alpha_1, \alpha_2 > -1$), then it suffices to postulate that $r_t \leq r_t^c$. When $x_1 \geq 0$ and $x_2 \geq 0$, it is not clear whether the inequalities $P^c(x_2, -A, -C) \leq P^r \leq P^h(x_1, A, C)$ are valid. Indeed, Proposition 3.5 only shows that they are valid under the assumption that $x_1 = x_2 = 0$.

Remark 3.4 As in Section 5.4 of [12], one can prove the monotonicity and stability properties of the price with respect to the initial endowment of each party. No difficulty arises in the case of the hedger's price. Since the counterparty's price $P^c(x_1, x_2, -A, -C)$ depends also on the hedger's initial endowment, the arguments used in [12] should be slightly modified.

3.2 Initial Endowments of Opposite Signs

So far, we worked under the assumption that the initial endowments of both parties are non-negative. We will now briefly examine the situation where $x_1 \geq 0$ and $x_2 \leq 0$. As in our previous work [12], the concept of a 'martingale measure' will now be specified in a more abstract way than in Assumption 3.2 by making reference to some auxiliary process, which is hereafter denoted as β .

Assumption 3.5 We postulate that:

(i) there exists a probability measure $\tilde{\mathbb{P}}^\beta$ equivalent to \mathbb{P} such that the processes $\tilde{S}^{i,\text{cld}}, i = 1, 2, \dots, d$, which are given by

$$d\tilde{S}_t^{i,\text{cld}} = dS_t^i + dA_t^i - \beta_t^i S_t^i dt \quad (3.21)$$

for some \mathbb{R}^d -valued, \mathbb{G} -adapted, bounded processes β satisfying $r^b \leq \beta^i$ for $i = 1, \dots, d$, are $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -continuous, square-integrable martingales and have the predictable representation property with respect to the filtration \mathbb{G} under $\tilde{\mathbb{P}}^\beta$,

(ii) there exists an $\mathbb{R}^{d \times d}$ -valued, \mathbb{G} -adapted process m such that

$$\langle \tilde{S}^{\text{cld}} \rangle_t = \int_0^t m_u m_u^* du \quad (3.22)$$

where mm^* is invertible and satisfies $mm^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$. Here σ is a d -dimensional square matrix of \mathbb{G} -adapted processes, which satisfies the ellipticity condition (3.7).

The following proposition establishes the no-arbitrage property of Bergman's model under the present assumptions. Since the proof of this result is very similar to the proof of Proposition 3.2 in [12], it is omitted.

Proposition 3.6 *If Assumption 3.5 holds, then Bergman's model is arbitrage-free for the hedger and the counterparty in respect of any initial endowments and any contract (A, C) .*

For the sake of comparison, we recall Proposition 3.3 from [14] concerning the case of exogenous collateral.

Proposition 3.7 *Let $x_1 \geq 0, x_2 \leq 0$ and Assumption 3.5 be valid. Then for any contract (A, C) such that $A^C \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ we have $P^h(x_1, A, C) = \tilde{Y}^{h,x_1} - C$ and $P^c(x_2, -A, -C) = \tilde{Y}^{c,x_2} - C$ where $(\tilde{Y}^{h,x_1}, \tilde{Z}^{h,x_1})$ is the unique solution to the following BSDE*

$$\begin{cases} d\tilde{Y}_t^{h,x_1} = \tilde{Z}_t^{h,x_1,*} d\tilde{S}_t^{\text{cld}} + G^h(t, x_1, \tilde{Y}_t^{h,x_1}, \tilde{Z}_t^{h,x_1}) dt + dA_t^C, \\ \tilde{Y}_T^{h,x_1} = 0, \end{cases}$$

and $(\tilde{Y}^{c,x_2}, \tilde{Z}^{c,x_2})$ is the unique solution to the following BSDE

$$\begin{cases} d\tilde{Y}_t^{c,x_2} = \tilde{Z}_t^{c,x_2,*} d\tilde{S}_t^{\text{cld}} + G^c(t, x_2, \tilde{Y}_t^{c,x_2}, \tilde{Z}_t^{c,x_2}) dt + dA_t^C, \\ \tilde{Y}_T^{c,x_2} = 0, \end{cases}$$

where the generators G^h and G^c are given by the following expressions

$$G^h(t, x_1, y, z) := \sum_{i=1}^d z^i \beta_t^i S_t^i - x_1 r_t^l B_t^l + r_t^l \left(y + x_1 B_t^l - z^* S_t \right)^+ - r_t^b \left(y + x_1 B_t^l - z^* S_t \right)^-$$

and

$$G^c(t, x_2, y, z) := \sum_{i=1}^d z^i \beta_t^i S_t^i + x_2 r_t^b B_t^b - r_t^l \left(-y + x_2 B_t^b + z^* S_t \right)^+ + r_t^b \left(-y + x_2 B_t^b + z^* S_t \right)^-.$$

The next result covers the case of hedger's collateral when the initial endowments x_1 and x_2 have opposite signs. Since its proof is analogous to that of Proposition 3.3, it is not presented here.

Proposition 3.8 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$, the hedger's ex-dividend price equals $P^h = \bar{Y}^1$ where (\bar{Y}^1, \bar{Z}^1) is the unique solution to the BSDE*

$$\begin{cases} d\bar{Y}_t^1 = \bar{Z}_t^{1,*} d\tilde{S}_t^{cld} + \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) dt + dA_t, \\ \bar{Y}_T^1 = 0, \end{cases} \quad (3.23)$$

with the generator \bar{f} given by

$$\begin{aligned} \bar{f}(t, x_1, y, z) = & \sum_{i=1}^d z^i \beta_t^i S_t^i - x_1 r_t^l B_t^l - r_t^c q(-y) \\ & + r_t^l \left(y + q(-y) + x_1 B_t^l - z^* S_t \right)^+ - r_t^b \left(y + q(-y) + x_1 B_t^l - z^* S_t \right)^- \end{aligned}$$

and the counterparty's ex-dividend price equals $P^c = \bar{Y}^2$ where (\bar{Y}^2, \bar{Z}^2) is the unique solution to the BSDE

$$\begin{cases} d\bar{Y}_t^2 = \bar{Z}_t^{2,*} d\tilde{S}_t^{cld} + \bar{g}(t, x_2, \bar{Y}_t^2, \bar{Z}_t^2) dt + dA_t, \\ \bar{Y}_T^2 = 0, \end{cases} \quad (3.24)$$

with the generator \bar{g} given by

$$\begin{aligned} \bar{g}(t, x_2, y, z) = & \sum_{i=1}^d z^i \beta_t^i S_t^i + x_2 r_t^b B_t^b - r_t^c q(-\bar{Y}_t^1) \\ & - r_t^l \left(-y - q(-\bar{Y}_t^1) + x_2 B_t^b + z^* S_t \right)^+ + r_t^b \left(-y - q(-\bar{Y}_t^1) + x_2 B_t^b + z^* S_t \right)^-. \end{aligned}$$

We are now in a position to analyze the range of fair bilateral prices when the initial endowments of counterparties are of opposite signs.

Proposition 3.9 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid.*

(i) *If $x_1 x_2 = 0$, then for any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ we have, for all $t \in [0, T]$,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.} \quad (3.25)$$

(ii) *Let r^l and r^b be deterministic and satisfy $r_t^l < r_t^b$ for all $t \in [0, T]$. Then (3.25) holds for all contracts (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ and all $t \in [0, T]$ if and only if $x_1 x_2 = 0$.*

Proof. (i) We consider solutions (\bar{Y}^1, \bar{Z}^1) and (\bar{Y}^2, \bar{Z}^2) to BSDEs (3.23) and (3.24) studied in Proposition 3.8 and we wish to apply the comparison theorem for BSDEs to show that $\bar{Y}^1 \geq \bar{Y}^2$. We claim that if $x_1 \geq 0$ and $x_2 \leq 0$, then

$$\delta := \bar{g}(t, x_2, \bar{Y}_t^1, \bar{Z}_t^1) - \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) \geq \max \{ -(r_t^b - r_t^l) x_1 B_t^l, (r_t^b - r_t^l) x_2 B_t^b \}. \quad (3.26)$$

Indeed, we have

$$\delta = x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-)$$

where we denote

$$\begin{aligned}\delta_1 &:= -\bar{Y}_t^1 - q(-\bar{Y}_t^1) + x_2 B_t^b + \bar{Z}_t^{1,*} S_t, \\ \delta_2 &:= \bar{Y}_t^1 + q(-\bar{Y}_t^1) + x_1 B_t^l - \bar{Z}_t^{1,*} S_t.\end{aligned}$$

From the postulated inequality $r_t^l \leq r_t^b$, it follows easily that

$$\delta \geq x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^l (\delta_1 + \delta_2) = (r_t^b - r_t^l) x_2 B_t^b$$

and

$$\delta \geq x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^b (\delta_1 + \delta_2) = -(r_t^b - r_t^l) x_1 B_t^l.$$

We have thus proven that (3.26) is valid. If $x_1 x_2 = 0$, then the right-hand side in (3.26) is non-negative. Hence $\delta \geq 0$ and thus, from the comparison theorem for BSDEs and the equality $(P^h, P^c) = (\bar{Y}^1, \bar{Y}^2)$ (see Proposition 3.8), we deduce that (3.25) is satisfied for every $t \in [0, T]$.

(ii) We now assume that the interest rates r^l and r^b are deterministic and satisfy $r_t^l < r_t^b$ for all $t \in [0, T]$. If $x_1 x_2 \neq 0$, then the example from the proof of Proposition 5.4 in [12] gives a contract (A, C) with $q \equiv 0$, such that the inequality $P_0^c(x_2, -A, -C) > P_0^h(x_1, A, C)$, $\widehat{\mathbb{P}}^\beta$ -a.s., holds in the present framework, so that the set $\mathcal{R}_0^f(x_1, x_2)$ is empty. \square

4 Bergman's Model with Negotiated Collateral

Our objective in Sections 4 and 6 is to analyze the situation where the collateral amount C relies upon both the hedger's value $V^h := V(x_1, \varphi, A, C)$ and the counterparty's value $V^c := V(x_2, \tilde{\varphi}, -A, -C)$. Specifically, Assumption 3.1 is replaced by the following postulate in which the collateral amount may depend on the contract's valuation by both parties. For convenience, we then say that the collateral is *negotiated* by the two parties, the sense that both the choice of the collateral convention \hat{q} and the dynamic computation of the collateral amount C_t involve both parties of a contract, in general.

Assumption 4.1 The *negotiated collateral* C is given by

$$C_t = \hat{q}(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)) \quad (4.1)$$

where $\hat{q}: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a uniformly Lipschitz continuous function such that $\hat{q}(0, 0) = 0$.

The case of a negotiated collateral should be contrasted with the situation considered in the preceding section, where it was postulated that the collateral amount was set by one party only. Let us observe that the prices for both parties will now depend on the vector of initial endowments (x_1, x_2) , but we will keep writing $P^h(x_1, A, C)$ and $P^c(x_2, -A, -C)$ instead of $P^h(x_1, x_2, A, C)$ and $P^c(x_1, x_2, -A, -C)$, respectively. For $x_1 \geq 0$ and $x_2 \geq 0$, using the arguments from the proof of Proposition 3.3, we obtain

$$P^h = P^h(x_1, A, C) = V(x_1, \varphi, A, C) - x_1 B^l = V^h - x_1 B^l$$

and

$$P^c = P^c(x_2, -A, -C) = -V(x_2, \tilde{\varphi}, -A, -C) + x_2 B^l = -V^c + x_2 B^l.$$

Similarly, for $x_2 \leq 0$, we have

$$P^c = P^c(x_2, -A, -C) = -V(x_2, \tilde{\varphi}, -A, -C) + x_2 B^b = -V^c + x_2 B^b.$$

We conclude that the following equality is valid, for $x_1 \geq 0$ and an arbitrary x_2 ,

$$C_t = \hat{q}(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)) = \hat{q}(-P_t^h, -P_t^c). \quad (4.2)$$

Example 4.1 As a particular instance of equation (4.1), we can consider the convex collateralization given by $\hat{q}(y_1, y_2) = \alpha y_1 + (1 - \alpha)y_2$ for some $\alpha \in [0, 1]$, so that

$$C_t = \alpha(V_t^0(x) - V_t^h) + (1 - \alpha)(V_t^c - V_t^0(x)) = -(\alpha P_t^h + (1 - \alpha)P_t^c).$$

4.1 Fully-Coupled Pricing BSDE

The following result, which covers the case of non-negative initial endowments, is a rather straightforward extension of Proposition 3.3 and thus its proof is omitted. It is worth noting that the processes Y and Z are \mathbb{R}^2 -valued and $\mathbb{R}^{d \times 2}$ -valued, respectively.

Proposition 4.1 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.3 and 4.1 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^l)$, the hedger's and counterparty's ex-dividend prices satisfy $(P^h, P^c)^* = Y$ where the pair (Y, Z) solves the following two-dimensional, fully-coupled BSDE*

$$\begin{cases} dY_t = Z_t^* d\tilde{S}_t^{l, cld} + g(t, Y_t, Z_t) dt + d\bar{A}_t, \\ Y_T = 0, \end{cases} \quad (4.3)$$

where $g = (g^1, g^2)^*$, $\bar{A} = (A, A)^*$ and, for all $y = (y_1, y_2)^* \in \mathbb{R}^2$, $z = (z_1, z_2) \in \mathbb{R}^{d \times 2}$,

$$\begin{aligned} g^1(t, y, z) &= r_t^l (B_t^l)^{-1} z_1^* S_t - x_1 B_t^l r_t^l - r_t^c \hat{q}(-y_1, -y_2) \\ &\quad + r_t^l \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \right)^+ \\ &\quad - r_t^b \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \right)^- \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} g^2(t, y, z) &= r_t^l (B_t^l)^{-1} z_2^* S_t + x_2 B_t^l r_t^l - r_t^c \hat{q}(-y_1, -y_2) \\ &\quad - r_t^l \left(-y_2 - \hat{q}(-y_1, -y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \right)^+ \\ &\quad + r_t^b \left(-y_2 - \hat{q}(-y_1, -y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \right)^-. \end{aligned} \quad (4.5)$$

Obviously, the prices for both parties depend here on the vector (x_1, x_2) of initial endowments, so that the notation $P^h = P^h(x_1, x_2, A, C)$ and $P^c = P^c(x_1, x_2, -A, -C)$ would be more appropriate. However, for brevity, they will still be denoted as $P^h(x_1, A, C)$ and $P^c(x_2, -A, -C)$, respectively. The case of initial endowments of opposite signs is covered by the following proposition, which corresponds to Proposition 3.8.

Proposition 4.2 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.5 and 4.1 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$, the hedger's and counterparty's ex-dividend prices satisfy $(P^h, P^c)^* = \hat{Y}$ where the pair (\hat{Y}, \hat{Z}) solves the following two-dimensional fully-coupled BSDE*

$$\begin{cases} d\hat{Y}_t = \hat{Z}_t^* d\tilde{S}_t^{cld} + \hat{g}(t, \hat{Y}_t, \hat{Z}_t) dt + d\bar{A}_t, \\ \hat{Y}_T = 0, \end{cases} \quad (4.6)$$

where $\hat{g} = (\hat{g}^1, \hat{g}^2)^*$, $\bar{A} = (A, A)^*$ and, for all $y = (y_1, y_2)^* \in \mathbb{R}^2$ and $z = (z_1, z_2) \in \mathbb{R}^{d \times 2}$,

$$\begin{aligned} \hat{g}^1(t, y, z) &= \sum_{i=1}^d z_1^i \beta_t^i S_t^i - x_1 B_t^l r_t^l - r_t^c \hat{q}(-y_1, -y_2) \\ &\quad + r_t^l \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \right)^+ \\ &\quad - r_t^b \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \right)^- \end{aligned} \quad (4.7)$$

and

$$\begin{aligned}\widehat{g}^2(t, y, z) &= \sum_{i=1}^d z_2^i \beta_t^i S_t^i + x_2 B_t^b r_t^b - r_t^c \widehat{q}(-y_1, -y_2) \\ &\quad - r_t^l \left(-y_2 - \widehat{q}(-y_1, -y_2) + x_2 B_t^b + z_2^* S_t \right)^+ \\ &\quad + r_t^b \left(-y_2 - \widehat{q}(-y_1, -y_2) + x_2 B_t^b + z_2^* S_t \right)^-.\end{aligned}\tag{4.8}$$

Proof. Once again, the proof is similar to the proof of Proposition 3.3. We also use Theorem 3.2 in [13] to show the well-posedness of BSDEs (4.3) and (4.6). Although the BSDE studied in [13] is one-dimensional, it is clear that Theorem 3.2 in [13] can be easily extended to the multi-dimensional framework. \square

4.2 Backward Stochastic Viability Property

To obtain the range of fair bilateral prices in the case of the negotiated collateral, one needs to compare the two components of a solution to fully-coupled BSDEs (4.3) and (4.6). When these BSDE are driven by a general continuous martingale, this is a challenging open problem. Fortunately, in most commonly used financial models, the pricing BSDEs are in fact driven by a Brownian motion. Under this assumption, using the ideas from Hu and Peng [10] and the characterization for the backward stochastic viability property (BSVP) given by Buckdahn et al. [4], we will be able to compare the two one-dimensional components, Y^1 and Y^2 , of a unique solution to BSDE (4.3) by producing first a suitable version of component-wise comparison theorem (see Theorem 4.1 below).

Let us first recall the definition of the *backward stochastic viability property* (BSVP, for short), which was studied by Buckdahn et al. [4]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the filtration \mathbb{F} generated by a d -dimensional Brownian motion W . For any Euclidean space \mathcal{H} , we denote $L_{ad}^2(\Omega, C([0, T], \mathcal{H}))$ by the closed, linear subspace of \mathbb{F} -adapted processes of the space $L^2(\Omega, \mathcal{F}, \mathbb{P}, C([0, T], \mathcal{H}))$. Also, let $L_{ad}^2(\Omega \times (0, T), \mathcal{H})$ be the Hilbert space of \mathbb{F} -adapted and measurable processes X such that $\|X\|_2 = \left(\mathbb{E} \int_0^T |X_t|^2 dt \right)^{1/2} < \infty$. We now consider the following n -dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \tag{4.9}$$

where η is an \mathbb{R}^n -valued random variable and the generator h satisfies the following assumption.

Assumption 4.2 Let the mapping $h : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n$ satisfy:

- (i) \mathbb{P} -a.s., for all $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$, the process $(h(t, y, z))_{t \in [0, T]}$ is \mathbb{F} -adapted and the mapping $t \rightarrow h(t, y, z)$ is continuous,
- (ii) the function h is uniformly Lipschitzian with respect to (y, z) : there exists a constant $L \geq 0$ such that \mathbb{P} -a.s. for all $t \in [0, T]$ and $y, y' \in \mathbb{R}^n, z, z' \in \mathbb{R}^{n \times d}$

$$|h(t, y, z) - h(t, y', z')| \leq L(|y - y'| + |z - z'|),$$

- (iii) the random variable $\sup_{t \in [0, T]} |h(t, 0, 0)|^2$ is square-integrable under \mathbb{P} .

The following definition is due to Buckdahn et al. [4].

Definition 4.1 We say that BSDE (4.9) has the *backward stochastic viability property* (BSVP) in K if and only if: for any $U \in [0, T]$ and an arbitrary $\eta \in L^2(\Omega, \mathcal{F}_U, \mathbb{P}; K)$, the unique solution $(Y, Z) \in L_{ad}^2(\Omega, C([0, U], \mathbb{R}^n)) \times L_{ad}^2(\Omega \times (0, U), \mathbb{R}^{n \times d})$ to the BSDE (4.9) over time interval $[0, U]$, that is,

$$Y_t = \eta + \int_t^U h(s, Y_s, Z_s) ds - \int_t^U Z_s dW_s, \tag{4.10}$$

satisfies $Y_t \in K$ for all $t \in [0, U]$, \mathbb{P} -a.s.

For a non-empty, closed, convex set of $K \subset \mathbb{R}^n$, let $\Pi_K(y)$ be the projection of a point $y \in \mathbb{R}^n$ onto K and let $d_K(y)$ be the distance between y and K . The following result was established by Buckdahn et al. [4].

Proposition 4.3 *Let the generator h satisfy Assumption 4.2. Then BSDE (4.9) has the BSVP in K if and only if for any $t \in [0, T]$, $z \in \mathbb{R}^{n \times d}$ and $y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y , we have*

$$4\langle y - \Pi_K(y), h(t, \Pi_K(y), z) \rangle \leq \langle D^2 d_K^2(y) z, z \rangle + M d_K^2(y) \quad (4.11)$$

where $M > 0$ is a constant independent of (t, y, z) .

Motivated by results from Hu and Peng [10], we will show that Proposition 4.3 can be used to establish a convenient version of component-wise comparison theorem for the two-dimensional BSDE. Specifically, we prove the following theorem, in which we denote $Y = (Y^1, Y^2)^*$, $Z = (Z^1, Z^2)^*$ and

$$h(t, y, z) = (h^1(t, y^1, y^2, z^1, z^2), h^2(t, y^1, y^2, z^1, z^2))^*.$$

Theorem 4.1 *Consider the two-dimensional BSDE*

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \quad (4.12)$$

driven by the d -dimensional Brownian motion W , where the generator $h = (h^1, h^2)^*$ satisfies Assumption 4.2. The following statements are equivalent:

- (i) for any $U \in [0, T]$ and $\eta^1, \eta^2 \in L^2(\Omega, \mathcal{F}_U, \mathbb{P}, \mathbb{R})$ such that $\eta^1 \geq \eta^2$, the unique solution $(Y, Z) \in L_{ad}^2(\Omega, C([0, U], \mathbb{R}^2)) \times L_{ad}^2(\Omega \times (0, U), \mathbb{R}^{2 \times d})$ to (4.12) on $[0, U]$ satisfies $Y_t^1 \geq Y_t^2$ for all $t \in [0, U]$,
- (ii) the following inequality holds, for all $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

$$\begin{aligned} & -4y_1^- [h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ & \leq M|y_1^-|^2 + 2|z_1|^2 \mathbb{1}_{\{y_1 < 0\}}, \quad \mathbb{P} - a.s. \end{aligned} \quad (4.13)$$

Proof. Let us denote $\tilde{Y} = (Y^1 - Y^2, Y^2)^*$, $\tilde{Z} = (Z^1 - Z^2, Z^2)^*$, $\tilde{\eta} = (\eta^1 - \eta^2, \eta^2)^*$ and $\tilde{h}(t, y, z) = (\tilde{h}^1(t, y, z), \tilde{h}^2(t, y, z))^*$ where

$$\tilde{h}^1(t, y, z) := h^1(t, y_1 + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1 + y_2, y_2, z_1 + z_2, z_2)$$

and

$$\tilde{h}^2(t, y, z) := h^2(t, y_1 + y_2, y_2, z_1 + z_2, z_2).$$

Then statement (i) is equivalent to the following condition:

- (iii) for any date $U \in [0, T]$ and an arbitrary $\tilde{\eta} = (\tilde{\eta}^1, \tilde{\eta}^2)^*$ such that $\tilde{\eta}^1 \geq 0$, the unique solution (\tilde{Y}, \tilde{Z}) to the following BSDE over time interval $[0, U]$

$$\tilde{Y}_t = \tilde{\eta} + \int_t^U \tilde{h}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^U \tilde{Z}_s dW_s \quad (4.14)$$

satisfies $\tilde{Y}^1 \geq 0$. By applying Proposition 4.3 to BSDE (4.14) and the convex, closed set $K = \mathbb{R}_+ \times \mathbb{R}$, we see that (iii) is in turn equivalent to (ii), since (4.11) coincides with (4.13) in that case. \square

4.3 Initial Endowments of Equal Signs

In Sections 4.3 and 4.4, we work under Assumption 3.4, so that we deal with a diffusion model. For simplicity, we present here the case of one risky asset driven by the one-dimensional Brownian motion W but, in view of Theorem 4.1, an extension to the case of d risky assets driven by a d -dimensional Brownian motion is rather straightforward. Let us recall that the process a is given by equation (3.15).

Assumption 4.3 We postulate that the process a satisfies Novikov's condition (3.16), the process $(\sigma(\cdot, S))^{-1}$ and all the interest rates are continuous processes, and the process $(\sigma(\cdot, S))^{-1}S$ is bounded.

Since

$$d\tilde{S}_t^{t, \text{cld}} = (\mu(t, S_t) + \kappa(t, S_t) - r_t^l S_t) dt + \sigma(t, S_t) dW_t = \sigma(t, S_t)(a_t dt + dW_t),$$

the pricing BSDE (4.3) reduces to

$$\begin{cases} dY_t = Z_t \sigma(t, S_t) dW_t + (g(t, Y_t, Z_t) + \sigma(t, S_t) a_t Z_t) dt + d\bar{A}_t, \\ Y_T = 0, \end{cases}$$

or, equivalently,

$$\begin{cases} dY_t = Z_t dW_t + (g(t, Y_t, (\sigma(t, S_t))^{-1} Z_t) + a_t Z_t) dt + d\bar{A}_t, \\ Y_T = 0. \end{cases} \quad (4.15)$$

We first focus on the valuation and hedging of the collateralized European contingent claim (H_T, C) given by (3.18). Then (4.15) is equivalent to the following BSDE, for $t \in [0, T]$,

$$Y_t = \begin{pmatrix} -H_T \\ -H_T \end{pmatrix} - \int_t^T Z_s dW_s - \int_t^T (g(s, Y_s, (\sigma(s, S_s))^{-1} Z_s) + a_s Z_s) ds$$

with an additional jump at terminal date T , which ensures that $Y_T = 0$. It is thus clear that it suffices to examine the following BSDE on $[0, T]$

$$\begin{cases} dY_t = Z_t d\tilde{W}_t^l + g(t, Y_t, (\sigma(t, S_t))^{-1} Z_t) dt, \\ Y_T = (-H_T, -H_T)^*, \end{cases} \quad (4.16)$$

where \tilde{W}^l is a Brownian motion under the probability measure $\tilde{\mathbb{P}}^l$ defined by (3.17). We are now in a position to study the range of fair bilateral prices at time t for the European claim with negotiated collateral. Recall that in the present framework we have that $P^h(x_1, A, C) = P^h(x_1, x_2, A, C)$ and $P^c(x_2, -A, -C) = P^c(x_1, x_2, -A, -C)$.

Proposition 4.4 *Let $x_1 \geq 0, x_2 \geq 0$ and Assumptions 3.4, 4.1 and 4.3 be valid. Consider an arbitrary collateralized European claim (H_T, C) where $H_T \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}^l)$. Then we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -H_T, -C) \leq P_t^h(x_1, H_T, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.} \quad (4.17)$$

Proof. We write $\sigma^{-1} := (\sigma(t, S_t))^{-1}$. It is sufficient to check that the functions h^1 and h^2 , which are given by

$$h^1(t, y_1, y_2, z_1, z_2) := -g^1(t, y_1, y_2, \sigma^{-1} z_1, \sigma^{-1} z_2)$$

and

$$h^2(t, y^1, y_2, z_1, z_2) := -g^2(t, y_1, y_2, \sigma^{-1} z_1, \sigma^{-1} z_2),$$

where g^1 and g^2 are given by (4.4) and (4.5) with $d = 1$, respectively, satisfy Assumption 4.2 under $\tilde{\mathbb{P}}^l$ and condition (4.13). First, using the continuity of $(\sigma(\cdot, S))^{-1}$, g^1 and g^2 with respect to t , we know that for $y_1, y_2, z_1, z_2 \in \mathbb{R}$, the function $h^1(t, y_1, y_2, z_1, z_2)$ and $h^2(t, y_1, y_2, z_1, z_2)$ are also continuous with respect to t . Second, since the process $(\sigma(\cdot, S))^{-1}S$ is bounded and the function \hat{q} is uniformly Lipschitz continuous, it is obvious that $h^1(t, y_1, y_2, z_1, z_2)$ and $h^2(t, y_1, y_2, z_1, z_2)$ are uniformly Lipschitz continuous with respect to (y_1, y_2, z_1, z_2) . Moreover, from $\hat{q}(0, 0) = 0$ and $x_1, x_2 \geq 0$, we obtain

$$h^1(t, 0, 0, 0, 0) = h^2(t, 0, 0, 0, 0) = 0.$$

We conclude that Assumption 4.2 holds for h^1 and h^2 . Let us check condition (4.13) is valid as well. If we set

$$\delta_1 := y_1^+ + y_2 + \widehat{q}(-y_1^+ - y_2, -y_2) + x_1 B_t^l - (B_t^l)^{-1} \sigma^{-1}(z_1 + z_2) S_t$$

and

$$\delta_2 := -y_2 - \widehat{q}(-y_1^+ - y_2, -y_2) + x_2 B_t^l + (B_t^l)^{-1} \sigma^{-1} z_2 S_t,$$

then

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &= -g^1(t, y_1^+ + y_2, y_2, \sigma^{-1}(z_1 + z_2), \sigma^{-1} z_2) + g^2(t, y_1^+ + y_2, y_2, \sigma^{-1}(z_1 + z_2), \sigma^{-1} z_2) \\ &= -r_t^l (B_t^l)^{-1} \sigma^{-1} z_1 S_t + (x_1 + x_2) B_t^l r_t^l - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-). \end{aligned}$$

Since $r_t^l \leq r_t^b$, we have

$$\begin{aligned} r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) &\leq r_t^l (\delta_1^+ + \delta_2^+) - r_t^l (\delta_1^- + \delta_2^-) = r_t^l (\delta_1 + \delta_2) \\ &= r_t^l y_1^+ + (x_1 + x_2) B_t^l r_t^l - r_t^l (B_t^l)^{-1} \sigma^{-1} z_1 S_t \end{aligned}$$

and

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &= -r_t^l (B_t^l)^{-1} \sigma^{-1} z_1 S_t + (x_1 + x_2) B_t^l r_t^l - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-) \geq -r_t^l y_1^+. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & -4y_1^- [h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ &\leq 4r_t^l y_1^- y_1^+ = 0 \leq M|y_1^-|^2 + 2z_1^2 \mathbf{1}_{\{y_1 < 0\}}, \end{aligned}$$

which is the desired condition (4.13). \square

Let us now consider a more general contract A where the hedger receives cash flows H_1, H_2, \dots, H_k at times $0 < t_1 \leq t_2 \leq \dots \leq t_k \leq T$, so that

$$A_t - A_0 = \sum_{l=1}^k \mathbf{1}_{[t_l, T]}(t) H_l$$

where $H_l \in L^2(\Omega, \mathcal{F}_{t_l}, \widetilde{\mathbb{P}}^l)$. For conciseness, we denote this claim as (H, C) .

Proposition 4.5 *Let $x_1 \geq 0, x_2 \geq 0$ and Assumptions 3.4, 4.1 and 4.3 be valid. Then for any collateralized claim (H, C) where $H_l \in L^2(\Omega, \mathcal{F}_{t_l}, \widetilde{\mathbb{P}}^l)$ for $l = 1, 2, \dots, k$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -H, -C) \leq P_t^h(x_1, H, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

Proof. We first study the problem on $[t_k, T]$. Since $dA_t = 0$, it is just a special case of Proposition 4.4 (it suffices to take $H_T = 0$), we have that $P_t^c \leq P_t^h$ for all $t \in [t_k, T]$. Indeed, one can check directly that the equalities $P_t^c = P_t^h = 0$ hold for all $t \in [t_k, T]$.

We now consider the problem on $[t_{k-1}, t_k]$. Recall that $(P^h, P^c)^* = Y = (Y_1, Y_2)^*$ where (Y, Z) solves BSDE (4.15). From the first step, we know that $Y_{1, t_k} = Y_{2, t_k} = 0$. Let us consider BSDE (4.15) on $[t_{k-1}, t_k]$. Noticing that A only changes at time t_k and $\Delta A_{t_k} = H_k$, we obtain, for $s \in [t_{k-1}, t_k]$,

$$Y_s = \begin{pmatrix} -H_k \\ -H_k \end{pmatrix} - \int_s^{t_k} Z_t d\widetilde{W}_t^l - \int_s^{t_k} g(t, Y_t, \sigma_t^{-1} Z_t) dt$$

where $\sigma_t^{-1} := (\sigma(t, S_t))^{-1}$. So this is nothing else than just the pricing BSDE for European claim with maturity t_k and with receiving payoff H_k . Therefore, using Proposition 4.4, we have that for all $t \in [t_{k-1}, t_k]$, $Y_{2, t} \leq Y_{1, t}$ which yields $P_t^c \leq P_t^h$.

We can extend this inequality to $[t_{k-2}, t_{k-1})$. Indeed, for $s \in [t_{k-2}, t_{k-1})$,

$$Y_s = \begin{pmatrix} Y_{1,t_{k-1}} - H_{k-1} \\ Y_{2,t_{k-1}} - H_{k-1} \end{pmatrix} - \int_s^{t_{k-1}} Z_t d\widetilde{W}_t^l - \int_s^{t_{k-1}} g(t, Y_t, \sigma_t^{-1} Z_t) dt.$$

Since from the second step, we know $Y_{2,t_{k-1}} \leq Y_{1,t_{k-1}}$, using Theorem 4.1 and the proof of Proposition 4.4, we obtain $Y_{2,t} \leq Y_{1,t}$ for all $t \in [t_{k-2}, t_{k-1})$, which in turn yields $P_t^c \leq P_t^h$ for all $t \in [t_{k-2}, t_{k-1})$. By the backward induction, we conclude that (4.17) holds for every $t \in [0, T]$. \square

We also have the following result for the counterparty's price.

Proposition 4.6 *Let $x_1 \geq 0, x_2 \geq 0$ and Assumptions 3.4, 4.1 and 4.3 be valid. Consider an arbitrary contract (A, C) where $A - A_0$ is a non-positive (or bounded from above, so that $A - A_0 \leq M$ for some constant M), continuous, \mathbb{G} -adapted process such that $\mathbb{E}_{\widetilde{\mathbb{P}}^l}[\sup_{t \in [0, T]} |A_t|^2] < \infty$. Then we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

Proof. Recall that $\sigma_t^{-1} := (\sigma(t, S_t))^{-1}$. We have $(P^h, P^c)^* = Y = (Y_1, Y_2)^*$ where (Y, Z) solves BSDE (4.15). Let $\widetilde{Y} := Y - \overline{A} + \overline{A}_0$, where $\overline{A} = (A, A)^*$ and $\overline{A}_0 = (A_0, A_0)^*$, so that

$$\begin{cases} d\widetilde{Y}_t = Z_t d\widetilde{W}_t^l + g(t, \widetilde{Y}_t + \overline{A}_t - \overline{A}_0, \sigma_t^{-1} Z_t) dt, \\ \widetilde{Y}_T = -\overline{A}_T. \end{cases}$$

Similarly as in the proof of Proposition 4.4, we let

$$h^1(t, y_1, y_2, z_1, z_2) := -g^1(t, y_1 + A_t - A_0, y_2 + A_t - A_0, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

and

$$h^2(t, y^1, y_2, z_1, z_2) := -g^2(t, y_1 + A_t - A_0, y_2 + A_t - A_0, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2).$$

Since A is continuous and $\mathbb{E}_{\widetilde{\mathbb{P}}^l}[\sup_{t \in [0, T]} |A_t|^2] < \infty$, it is not hard to check that Assumption 4.2 is satisfied by h^1 and h^2 . Moreover, since $A - A_0 \leq 0$ (or $A - A_0 \leq M$), we have

$$\begin{aligned} & -4y_1^- [h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ & \leq 4r_t^l y_1^- (y_1^+ + A_t - A_0) \leq |M| |y_1^-|^2 + 2z_1^2 \mathbb{1}_{\{y_1 < 0\}}. \end{aligned}$$

To complete the proof, it suffices to use Theorem 4.1. \square

Remark 4.1 For a contract (A, C) with a more general process A , we may not have similar results. This is because that in (4.15), a general cash flow \overline{A} may destroy the viability property. However, by mixing the two kinds of special contracts introduced in Propositions 4.5 and 4.6, we can construct the following class of contracts: for $0 < t_1 \leq t_2 \leq \dots \leq t_k \leq T$, and processes $H_l(t)$, $l = 1, \dots, k$ defined on $[t_l, T]$,

$$A_t - A_0 = \sum_{l=1}^k \mathbb{1}_{[t_l, T]}(t) H_l(t)$$

where, for $l = 1, 2, \dots, k$, the process $H_l(t)$, $t \in [t_l, T]$, satisfies one of the following conditions:

- (i) H_l is a continuous, \mathbb{G} -adapted process, $H_l \leq M$ and $\mathbb{E}_{\widetilde{\mathbb{P}}^l}[\sup_{t \in [t_l, T]} |H_l(t)|^2] < \infty$,
- (ii) $H_l(t) = H_l$ for all $t \in [t_l, T]$, where the random variable $H_l \in L^2(\Omega, \mathcal{F}_{t_l}, \mathbb{P})$.

By combining the statements and proofs of Propositions 4.5 and 4.6, one can show that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ for the contract (A, C) satisfying (i)–(ii) is non-empty almost surely.

4.4 Initial Endowments of Opposite Signs

We only consider here the case of a collateralized European contingent claim (H_T, C) , but similar results hold for two special kinds of contracts introduced in Propositions 4.5 and 4.6. We work under Assumptions 3.4 and 3.5 and we denote

$$b_t := (\sigma(t, S_t))^{-1} (\mu(t, S_t) + \kappa(t, S_t) - \beta_t S_t). \quad (4.18)$$

Assumption 4.4 We postulate that the process b satisfies Novikov's condition (3.16), the processes $(\sigma(\cdot, S))^{-1}$, β and all interest rates are continuous processes and the process $(\sigma(\cdot, S))^{-1} S$ is bounded.

We observe that

$$d\tilde{S}_t^{\text{cld}} = (\mu(t, S_t) + \kappa(t, S_t) - \beta_t S_t) dt + \sigma(t, S_t) dW_t = \sigma(t, S_t) (b_t dt + dW_t) = \sigma(t, S_t) d\tilde{W}_t^\beta$$

where $d\tilde{W}_t^\beta := dW_t + b_t dt$. Let us define the probability measure $\tilde{\mathbb{P}}^\beta$ by setting

$$\frac{d\tilde{\mathbb{P}}^\beta}{d\mathbb{P}} = \exp \left\{ - \int_0^T b_t dW_t - \frac{1}{2} \int_0^T |b_t|^2 dt \right\}.$$

From the Girsanov theorem, the process \tilde{W}^β is the Brownian motion under $\tilde{\mathbb{P}}^\beta$ and thus \tilde{S}^{cld} is a $(\tilde{\mathbb{P}}^\beta, \mathbb{G})$ -(local) martingale with the quadratic variation $\langle \tilde{S}^{\text{cld}} \rangle_t = \int_0^t |\sigma(u, S_u)|^2 du$. Moreover, since the process $(\sigma(\cdot, S))^{-1} S$ is bounded, Assumption 3.5 holds. We conclude that the model is arbitrage free under $\tilde{\mathbb{P}}^\beta$ (see Proposition 3.6).

Under the present framework, BSDE (4.6) can be represented as follows

$$\begin{cases} dY_t = Z_t \sigma(t, S_t) dW_t + (\hat{g}(t, Y_t, Z_t) + \sigma(t, S_t) b_t Z_t) dt + d\bar{A}_t, \\ Y_T = 0. \end{cases}$$

As in Section 4.3, it is thus sufficient to examine the following BSDE on $[0, T]$

$$\begin{cases} dY_t = Z_t d\tilde{W}_t^\beta + \hat{g}(t, Y_t, (\sigma(t, S_t))^{-1} Z_t) dt, \\ Y_T = (-H_T, -H_T)^*. \end{cases}$$

Proposition 4.7 *Let $x_1 \geq 0$, $x_2 \leq 0$ be such that $x_1 x_2 = 0$. If Assumptions 3.4, 3.5, 4.1 and 4.4 are satisfied, then for any collateralized European claim (H_T, C) such that $H_T \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}^\beta)$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -H_T, -C) \leq P_t^h(x_1, H_T, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

Proof. Let $\sigma_t^{-1} := (\sigma(t, S_t))^{-1}$. It suffices to check that the functions

$$h^1(t, y_1, y_2, z_1, z_2) := -\hat{g}^1(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

and

$$h^2(t, y^1, y_2, z_1, z_2) := -\hat{g}^2(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

satisfy Assumption 4.2 and condition (4.13), where \hat{g}^1 and \hat{g}^2 are given by (4.7) and (4.8) with $d = 1$, respectively. First, from the continuity of β , σ^{-1} , \hat{g}^1 and \hat{g}^2 with respect to t , we deduce that for $y_1, y_2, z_1, z_2 \in \mathbb{R}$, the functions $h^1(t, y_1, y_2, z_1, z_2)$ and $h^2(t, y_1, y_2, z_1, z_2)$ are also continuous with respect to t . Second, since $\sigma^{-1} S$ is bounded and \hat{q} is uniformly Lipschitz continuous, it is clear that $h^1(t, y_1, y_2, z_1, z_2)$ and $h^2(t, y_1, y_2, z_1, z_2)$ are uniformly Lipschitz continuous with respect to (y_1, y_2, z_1, z_2) . Moreover, from $\hat{q}(0, 0) = 0$ and $x_1 \geq 0$, and $x_2 \leq 0$, we have that $h^1(t, 0, 0, 0, 0) =$

$h^2(t, 0, 0, 0, 0) = 0$. We thus see that Assumption 4.2 holds for h^1 and h^2 . Finally, let us check that condition (4.13) is met as well. To this end, we set

$$\delta_1 := y_1^+ + y_2 + \widehat{q}(-y_1^+ - y_2, -y_2) + x_1 B_t^l - \sigma_t^{-1}(z_1 + z_2) S_t$$

and

$$\delta_2 := -y_2 - \widehat{q}(-y_1^+ - y_2, -y_2) + x_2 B_t^b + \sigma_t^{-1} z_2 S_t.$$

Then

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &= -\widehat{g}^1(t, y_1^+ + y_2, y_2, \sigma_t^{-1}(z_1 + z_2), \sigma_t^{-1} z_2) + \widehat{g}^2(t, y_1^+ + y_2, y_2, \sigma_t^{-1}(z_1 + z_2), \sigma_t^{-1} z_2) \\ &= -\sigma_t^{-1} \beta_t (z_1 + z_2) S_t + x_1 B_t^l r_t^l + r_t^c \widehat{q}(-y_1^+ - y_2, -y_2) - r_t^l \delta_1^+ + r_t^b \delta_1^- \\ &\quad + \sigma_t^{-1} \beta_t z_2 S_t + x_2 B_t^b r_t^b - r_t^c \widehat{q}(-y_1^+ - y_2, -y_2) - r_t^l \delta_2^+ + r_t^b \delta_2^- \\ &= -\sigma_t^{-1} \beta_t z_1 S_t + x_1 B_t^l r_t^l + x_2 B_t^b r_t^b - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-). \end{aligned}$$

Since $r_t^l \leq r_t^b$, we have

$$\begin{aligned} & r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) \leq \min \{ r_t^l (\delta_1 + \delta_2), r_t^b (\delta_1 + \delta_2) \} \\ &= \min \{ r_t^l y_1^+ + x_1 B_t^l r_t^l + x_2 B_t^b r_t^b - r_t^l \sigma_t^{-1} z_1 S_t, r_t^b y_1^+ + x_1 B_t^l r_t^b + x_2 B_t^b r_t^b - r_t^b \sigma_t^{-1} z_1 S_t \}. \end{aligned}$$

Thus

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \geq -\sigma_t^{-1} \beta_t z_1 S_t \\ &+ \max \{ -r_t^l y_1^+ + x_2 B_t^b r_t^b - x_2 B_t^b r_t^l + r_t^l \sigma_t^{-1} z_1 S_t, -r_t^b y_1^+ + x_1 B_t^l r_t^l - x_1 B_t^l r_t^b + r_t^b \sigma_t^{-1} z_1 S_t \}. \end{aligned}$$

We also have that

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &\geq -r_t^l y_1^+ + \sigma_t^{-1} S_t (r_t^l - \beta_t) z_1 + x_2 B_t^b (r_t^b - r_t^l). \end{aligned}$$

Consequently, if $x_2 = 0$ then, using the boundedness of processes β , r^l and $\sigma^{-1} S$, we obtain

$$\begin{aligned} & -4y_1^- [h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ &\leq 4r_t^l y_1^- y_1^+ - 4y_1^- z_1 \sigma_t^{-1} S_t (r_t^l - \beta_t) = -4y_1^- z_1 \sigma_t^{-1} S_t (r_t^l - \beta_t) \\ &\leq M |y_1^-|^2 + 2z_1^2 \mathbf{1}_{\{y_1 < 0\}}, \end{aligned}$$

which is the desired inequality (4.13). The same inequality can be obtained when $x_1 = 0$. \square

Remark 4.2 Let us consider a more general class of contracts considered in Section 4.3. We now assume that the interest rates r^l and r^b are deterministic and satisfy $r_t^l < r_t^b$ for all $t \in [0, T]$. Using the example studied in the proof of Proposition 5.4 in [12], for every model for risky assets we see that for every contract (A, C) considered in Section 4.3, for all $t \in [0, T]$,

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^\beta - \text{a.s.}$$

if and only if $x_1 x_2 = 0$.

Remark 4.3 As was explained in Section 3.1, the price $P^h(x_1, A, C)$ (resp., $P^c(x_2, -A, -C)$) indeed should be $P^h(x_1, x_2, A, C)$ (resp., $P^c(x_1, x_2, -A, -C)$), meaning that the hedger's and the counterparty's price depend on both initial endowments x_1 and x_2 . Using the comparison theorem for multi-dimensional BSDE (see Hu and Peng [10]), one may attempt to show the monotonicity of prices with respect to the initial endowment (for related results, see Section 5.4 in [12]).

5 Model with Partial Netting and Hedger's Collateral

In Sections 5 and 6, we consider the model with partial netting and full rehypothecation of the cash collateral. For a detailed description of this modeling framework, the reader is referred to [2, 12]. Our aim is to show that the methodology developed in preceding sections can be applied to this setup, albeit with possibly different conclusions regarding the properties of unilateral and bilateral prices. Since the proofs of some results are very similar to the proofs of their counterparts in Bergman's model, they are omitted.

From Lemma 2.1 and Lemma 2.2 of [12], we know that for a self-financing trading strategy

$$\varphi = (\xi^1, \dots, \xi^d, \varphi^l, \varphi^b, \varphi^{1,b}, \varphi^{2,b}, \dots, \varphi^{d,b}, \eta),$$

the processes $Y^l := (B^l)^{-1}V^p(x, \varphi, A, C)$ and $Z^{l,i} = \xi^i$, $i = 1, 2, \dots, d$ satisfy

$$dY_t^l = \sum_{i=1}^d Z_t^{l,i} d\tilde{S}_t^{i,l,\text{cld}} + G_l(t, Y_t^l, Z_t^l) dt + dA_t^{C,l} \quad (5.1)$$

where the generator G_l equals, for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} G_l(t, y, z) = & (B_t^l)^{-1} \sum_{i=1}^d r_t^l z^i S_t^i - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - r_t^l y \\ & + (B_t^l)^{-1} \left(r_t^l \left(y B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left(y B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^- \right). \end{aligned}$$

Similarly, the processes $Y^b := (B^b)^{-1}V^p(x, \varphi, A, C)$ and $Z^{b,i} = \xi^i$, $i = 1, 2, \dots, d$ satisfy

$$dY_t^b = \sum_{i=1}^d Z_t^{b,i} d\tilde{S}_t^{i,b,\text{cld}} + G_b(t, Y_t^b, Z_t^b) dt + dA_t^{C,b} \quad (5.2)$$

where, for all $(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$\begin{aligned} G_b(t, y, z) = & (B_t^b)^{-1} \sum_{i=1}^d r_t^b z^i S_t^i - (B_t^b)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - r_t^b y \\ & + (B_t^b)^{-1} \left(r_t^l \left(y B_t^b + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ - r_t^b \left(y B_t^b + \sum_{i=1}^d (z^i S_t^i)^- \right)^- \right). \end{aligned}$$

Throughout Section 5, we work under Assumption 3.1 of hedger's collateral.

5.1 Initial Endowments of Equal Signs

We first examine the case where the initial endowments satisfy $x_1 \geq 0$ and $x_2 \geq 0$. We notice that in such case, under Assumption 3.2 (or Assumption 3.3) the partial netting model is arbitrage-free with respect to any contract (A, C) for the hedger and the counterparty (see Proposition 3.1 in [12]). Using Propositions 4.1 and 4.2 in [12], we can establish the following proposition, which corresponds to Proposition 3.3 in the present work.

Proposition 5.1 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.1 and 3.3 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\mathbb{P}^l)$, the hedger's ex-dividend price equals $P^h := P^h(x_1, A, C) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE*

$$\begin{cases} dY_t^1 = Z_t^{1,*} d\tilde{S}_t^{l,\text{cld}} + f_1(t, x_1, Y_t^1, Z_t^1) dt + dA_t, \\ Y_T^1 = 0, \end{cases} \quad (5.3)$$

with the generator f_l given by

$$\begin{aligned} f_l(t, x_1, y, z) = & r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - x_1 B_t^l r_t^l - r_t^c q(-y) \\ & + r_t^l \left(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (z^i S_t^i)^- \right)^+ \\ & - r_t^b \left(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (z^i S_t^i)^- \right)^- \end{aligned} \quad (5.4)$$

and the counterparty's ex-dividend price equals $P^c := P^c(x_2, -A, -C) = Y^2$ where (Y^2, Z^2) is the unique solution to the BSDE

$$\begin{cases} dY_t^2 = Z_t^{2,*} d\tilde{S}_t^{l, cld} + g_l(t, x_2, Y_t^2, Z_t^2) dt + dA_t, \\ Y_T^2 = 0, \end{cases} \quad (5.5)$$

with the generator g_l given by

$$\begin{aligned} g_l(t, x_2, y, z) = & r_t^l (B_t^l)^{-1} z^* S_t + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ + x_2 B_t^l r_t^l - r_t^c q(-Y_t^1) \\ & - r_t^l \left(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z^i S_t^i)^- \right)^+ \\ & + r_t^b \left(-y - q(-Y_t^1) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z^i S_t^i)^- \right)^-. \end{aligned} \quad (5.6)$$

As in Bergman's model, if the convention of hedger's collateral is postulated, then we have $P^h = P^h(x_1, A, C)$ and $P^c = P^c(x_1, x_2, -A, -C)$, but we still denote the counterparty's price as $P^c(x_2, -A, -C)$. We are in a position to study the range of fair bilateral prices.

Proposition 5.2 *Let $x_1 \geq 0, x_2 \geq 0$ and Assumptions 3.1 and 3.3 be valid. Then for any contract (A, C) such that $A \in \mathcal{A}(\mathbb{P}^l)$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.} \quad (5.7)$$

Proof. It is enough to show that $g_l(t, x_2, Y^1, Z^1) \geq f_l(t, x_1, Y^1, Z^1)$, $\tilde{\mathbb{P}}^l \otimes \ell - \text{a.e.}$ We denote

$$\begin{aligned} \delta &:= g_l(t, x_2, Y^1, Z^1) - f_l(t, x_1, Y^1, Z^1) \\ &= r_t^l B_t^l (x_1 + x_2) + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} |Z_t^{1,i} S_t^i| - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-) \end{aligned}$$

where

$$\delta_1 := -Y_t^1 - q(-Y_t^1) + B_t^l x_2 + (B_t^l)^{-1} \sum_{i=1}^d (-Z_t^{1,i} S_t^i)^-$$

and

$$\delta_2 := Y_t^1 + q(-Y_t^1) + B_t^l x_1 + (B_t^l)^{-1} \sum_{i=1}^d (Z_t^{1,i} S_t^i)^-.$$

Since $r^l \leq r^b$ and $r^l \leq r^{i,b}$, we obtain

$$\begin{aligned} \delta &\geq r_t^l B_t^l (x_1 + x_2) + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} |Z_t^{1,i} S_t^i| - r_t^l (\delta_1 + \delta_2) \\ &\geq (B_t^l)^{-1} \sum_{i=1}^d (r_t^{i,b} - r^l) |Z_t^{1,i} S_t^i| \geq 0, \end{aligned}$$

which completes the proof. \square

5.1.1 Model with an Uncertain Money Market Rate

We study here the case of initial endowments satisfying $x_1 \geq 0$ and $x_2 \geq 0$, but results for the case where $x_1 \geq 0$ and $x_2 \leq 0$ are similar. Let us select an arbitrary \mathbb{G} -adapted interest rate process satisfying

$$r_t \in [r_t^l, r_t^b] \quad \text{for every } t \in [0, T]. \quad (5.8)$$

We now consider the market model with the single money market rate r in which the hedger and counterparty have the same ex-dividend price P^r independent of their respective initial endowments. The price $P^r = Y$ can be found by solving the BSDE

$$\begin{cases} dY_t = Z_t^* d\tilde{S}_t^{\text{cld}} + f(t, Y_t, Z_t) dt + dA_t, \\ Y_T = 0, \end{cases} \quad (5.9)$$

where the generator f equals

$$f(t, y, z) = r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - (B_t^l)^{-1} \sum_{i=1}^d r_t (z^i S_t^i)^- - r_t^c q(-y) + r_t (y + q(-y)).$$

Similarly to Proposition 3.5, we have the following result under Assumption 3.1.

Proposition 5.3 *For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^l)$, the unique no-arbitrage price in the market model with the money market rate r satisfies $P^r \leq P^h(0, A, C)$, $\tilde{\mathbb{P}}^l$ -a.s. If $x_1 = x_2 = 0$ and in addition, the function q in (3.5) satisfies $(r_t - r_t^c)(q(y_1) - q(y_2)) \leq 0$ for all $y_1 \geq y_2$, then also $P^c(0, -A, -C) \leq P^r$, $\tilde{\mathbb{P}}^l$ -a.s.*

5.2 Initial Endowments of Opposite Signs

Let us now consider the case where $x_1 \geq 0$ and $x_2 \leq 0$. We now postulate that $r^b \leq r^{i,b}$ and Assumption 3.5 holds with $r^b \leq \beta^i \leq r^{i,b}$. From Proposition 3.2 in [12], we know that the partial netting model is arbitrage-free for both the hedger and the counterparty in respect of any contract (A, C) and arbitrary initial endowments. Using Proposition 5.3 in [12] and argument similar as in the proof of Proposition 3.8, one can prove the following propositions.

Proposition 5.4 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$, the hedger's ex-dividend price equals $P^h = \bar{Y}^1$ where the pair (\bar{Y}^1, \bar{Z}^1) is the unique solution to the BSDE*

$$\begin{cases} d\bar{Y}_t^1 = \bar{Z}_t^{1,*} d\tilde{S}_t^{\text{cld}} + \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) dt + dA_t, \\ \bar{Y}_T^1 = 0, \end{cases} \quad (5.10)$$

where

$$\begin{aligned} \bar{f}(t, x_1, y, z) &= \sum_{i=1}^d z^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ - x_1 r_t^l B_t^l - r_t^c q(-y) \\ &\quad + r_t^l \left(y + q(-y) + x_1 B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^+ \\ &\quad - r_t^b \left(y + q(-y) + x_1 B_t^l + \sum_{i=1}^d (z^i S_t^i)^- \right)^- \end{aligned}$$

and the counterparty's ex-dividend price equals $P^c = \bar{Y}^c$ where the pair (\bar{Y}^c, \bar{Z}^c) is the unique solution to the BSDE

$$\begin{cases} d\bar{Y}_t^c = \bar{Z}_t^{c,*} d\tilde{S}_t^{\text{cld}} + \bar{g}(t, x_2, \bar{Y}_t^c, \bar{Z}_t^c) dt + dA_t, \\ \bar{Y}_T^c = 0, \end{cases} \quad (5.11)$$

where

$$\begin{aligned}\bar{g}(t, x_2, y, z) = & \sum_{i=1}^d z^i \beta_t^i S_t^i + \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ + x_2 r_t^b B_t^b - r_t^c q(-\bar{Y}_t^1) \\ & - r_t^l \left(-y - q(-\bar{Y}_t^1) + x_2 B_t^b + \sum_{i=1}^d (-z^i S_t^i)^- \right)^+ \\ & + r_t^b \left(-y - q(-\bar{Y}_t^1) + x_2 B_t^b + \sum_{i=1}^d (-z^i S_t^i)^- \right)^-.\end{aligned}$$

The following result shows that the range of fair bilateral prices is non-empty provided that $x_1 x_2 = 0$. Otherwise, one can produce an example of a model in which this range is empty.

Proposition 5.5 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid.*

(i) *If $x_1 x_2 = 0$, then for any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ we have, for every $t \in [0, T]$*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.} \quad (5.12)$$

(ii) *Let r^l and r^b be deterministic and satisfy $r_t^l < r_t^b$ for all $t \in [0, T]$. Then (5.12) holds for all contracts (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ and all $t \in [0, T]$ if and only if $x_1 x_2 = 0$.*

Proof. (i) Assume that $x_1 \geq 0$ and $x_2 \leq 0$. We will show that

$$\begin{aligned}\delta &:= \bar{g}(t, x_2, \bar{Y}^1, \bar{Z}^1) - \bar{f}(t, x_1, \bar{Y}^1, \bar{Z}^1) \\ &\geq \max \left\{ -(r_t^b - r_t^l) x_1 B_t^l + \sum_{i=1}^d (r_t^{i,b} - r_t^b) |\bar{Z}_t^{1,i} S_t^i|, (r_t^b - r_t^l) x_2 B_t^b + \sum_{i=1}^d (r_t^{i,b} - r_t^l) |\bar{Z}_t^{1,i} S_t^i| \right\}.\end{aligned}$$

Indeed, we have

$$\delta = x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-)$$

where

$$\delta_1 := -\bar{Y}_t^1 - q(-\bar{Y}_t^1) + x_2 B_t^b + \sum_{i=1}^d (-\bar{Z}_t^{1,i} S_t^i)^-, \quad \delta_2 := \bar{Y}_t^1 + q(-\bar{Y}_t^1) + x_1 B_t^l + \sum_{i=1}^d (\bar{Z}_t^{1,i} S_t^i)^-.$$

From $r_t^l \leq r_t^b$, it follows that

$$\delta \geq \sum_{i=1}^d r_t^{i,b} |z^i S_t^i| + x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^l (\delta_1 + \delta_2) = (r_t^b - r_t^l) x_2 B_t^b + \sum_{i=1}^d (r_t^{i,b} - r_t^l) |\bar{Z}_t^{1,i} S_t^i|$$

and

$$\delta \geq \sum_{i=1}^d r_t^{i,b} |z^i S_t^i| + x_1 r_t^l B_t^l + x_2 r_t^b B_t^b - r_t^b (\delta_1 + \delta_2) = -(r_t^b - r_t^l) x_1 B_t^l + \sum_{i=1}^d (r_t^{i,b} - r_t^b) |\bar{Z}_t^{1,i} S_t^i|.$$

We have thus shown that

$$\delta \geq \max \left\{ -(r_t^b - r_t^l) x_1 B_t^l + \sum_{i=1}^d (r_t^{i,b} - r_t^b) |\bar{Z}_t^{1,i} S_t^i|, (r_t^b - r_t^l) x_2 B_t^b + \sum_{i=1}^d (r_t^{i,b} - r_t^l) |\bar{Z}_t^{1,i} S_t^i| \right\}.$$

If $x_1 x_2 = 0$, then using $r_t^{i,b} \geq r_t^b \geq r_t^l$, it is easy to check that the right-hand side of the above inequality is non-negative. Hence $\delta \geq 0$ and thus, from the comparison theorem for BSDEs and Proposition 3.8, we deduce that inequality (5.12) holds for every $t \in [0, T]$.

(ii) If $x_1 x_2 \neq 0$, then the example from the proof of Proposition 5.4 in [12] gives a contract (A, C) with $q \equiv 0$, such that the inequality

$$P_0^c(x_2, -A, -C) > P_0^h(x_1, A, C), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

holds in the present framework, so that $\mathcal{R}_0^p(x_1, x_2)$ is non-empty almost surely. \square

Remark 5.1 If $x_1 x_2 < 0$ then, from the above proposition, we know that for some contracts (A, C) , we have $P_{\hat{t}}^c(x_2, -A, -C) > P_{\hat{t}}^h(x_1, A, C)$ for some $\hat{t} \in [0, T]$. As in [12], for some special contracts of (A, C) , the inequality $P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C)$ holds for all $t \in [0, T]$. We will discuss in next subsection.

5.2.1 Contracts with Monotone Cash Flows

We continue the study of the case where $x_1 \geq 0$ and $x_2 \leq 0$. Motivated by [12], we will show that for some special contracts (A, C) , inequality (5.12) holds for all $t \in [0, T]$.

Assumption 5.1 The following conditions are satisfied by a contract (A, C) :

- (i) the process $A - A_0$ is decreasing and belongs to the class $\mathcal{A}(\tilde{\mathbb{P}}^\beta)$,
- (ii) the collateral C is given by (3.5) with the function q satisfying $y + q(-y) \geq 0$ for all $y \geq 0$.

Condition (ii) holds, for instance, when $q(y) = (1 + \alpha_1)y^+ - (1 + \alpha_2)y^-$ for some haircut processes α_1, α_2 such that $\alpha_2 \leq 0$ which means that when the hedger posts collateral then the cash amount never exceeds the full collateral. Indeed, q is obviously uniformly Lipschitz continuous and $q(0) = 0$. Moreover, we have, for all $y \geq 0$,

$$y + q(-y) = y - (1 + \alpha_2)y = -\alpha_2 y \geq 0.$$

To emphasize the important role of the function q , we will sometimes write $P_t^h(x_1, A, q)$ and $P_t^c(x_1, -A, -q)$ instead of $P_t^h(x_1, A, C)$ and $P_t^c(x_2, -A, -C)$, respectively.

Remark 5.2 In the case of Bergman's model, we were unable to prove that the range of fair bilateral prices is non-empty using the method employed to establish the next result. This shows once again that the properties of prices depend on specific features of a market model at hand.

Proposition 5.6 Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid. If a contract (A, C) satisfies Assumption 5.1, then the inequality $P_t^c(x_2, -A, -q) \leq P_t^h(x_1, A, q)$ holds for every $t \in [0, T]$.

Proof. We already know that the pair $(P_t^h, \tilde{Z}_t^{h, x_1})$ solves BSDE (5.10), whereas the pair $(P_t^c, \tilde{Z}_t^{c, x_2})$ solves BSDE (5.11). Noticing that $\bar{f}(t, x_1, 0, 0) = 0$ and $A - A_0$ is a decreasing process, from the comparison theorem for BSDEs, we obtain $P^h = \bar{Y}^1 \geq 0$. Therefore, from $x_1 \geq 0$ and

$$y + q(-y) \geq 0 \text{ for all } y \geq 0,$$

we get

$$\begin{aligned} \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) &= \sum_{i=1}^d \bar{Z}_t^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{Z}_t^i S_t^i)^+ - x_1 r_t^l B_t^l - r_t^c q(-\bar{Y}_t^1) \\ &\quad + r_t^l (\bar{Y}_t^1 + q(-\bar{Y}_t^1) + x_1 B_t^l + \sum_{i=1}^d (\bar{Z}_t^i S_t^i)^-) \\ &= \sum_{i=1}^d \bar{Z}_t^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{Z}_t^i S_t^i)^+ - r_t^c q(-\bar{Y}_t^1) \\ &\quad + r_t^l (\bar{Y}_t^1 + q(-\bar{Y}_t^1) + \sum_{i=1}^d (\bar{Z}_t^i S_t^i)^-). \end{aligned}$$

Since

$$\begin{aligned} \bar{g}(t, x_2, y, z) &\geq \sum_{i=1}^d z^i \beta_t^i S_t^i + \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ + x_2 r_t^b B_t^b - r_t^c q(-\bar{Y}_t^1) \\ &\quad - r_t^b (-y - q(-\bar{Y}_t^1) + x_2 B_t^b + \sum_{i=1}^d (-z^i S_t^i)^-) \\ &= \sum_{i=1}^d z^i \beta_t^i S_t^i + \sum_{i=1}^d r_t^{i,b} (-z^i S_t^i)^+ - r_t^c q(-\bar{Y}_t^1) \\ &\quad - r_t^b (-y - q(-\bar{Y}_t^1) + \sum_{i=1}^d (-z^i S_t^i)^-), \end{aligned}$$

we have that

$$\begin{aligned} &\bar{g}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) - \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) \\ &\geq \sum_{i=1}^d r_t^{i,b} |\bar{Z}_t^i S_t^i| + (r_t^b - r_t^l) (\bar{Y}_t^1 + q(-\bar{Y}_t^1)) - r_t^l \sum_{i=1}^d (\bar{Z}_t^i S_t^i)^- - r_t^b \sum_{i=1}^d (-\bar{Z}_t^i S_t^i)^- \\ &= (r_t^b - r_t^l) (\bar{Y}_t^1 + q(-\bar{Y}_t^1)) + \sum_{i=1}^d (r_t^{i,b} - r_t^l) (\bar{Z}_t^i S_t^i)^- + \sum_{i=1}^d (r_t^{i,b} - r_t^b) (-\bar{Z}_t^i S_t^i)^-. \end{aligned}$$

In view of the inequalities $r^{i,b} \geq r^b \geq r^l$, $\bar{Y}^1 \geq 0$ and $y + q(-y) \geq 0$ for all $y \geq 0$, we conclude that

$$\bar{g}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) - \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) \geq 0$$

and thus the comparison theorem for BSDEs yields the desired inequality. \square

5.3 Price Independence of the Hedger's Initial Endowment

Our next goal is to demonstrate that for a certain class of contracts the hedger's price in the model with partial netting is independent of the initial endowment x_1 . The financial interpretation of Proposition 5.7 is that the hedger will never need to borrow cash from the account B^b for hedging purposes and thus the actual level of his non-negative initial endowment is immaterial for his pricing problem. It is thus clear that a similar result will not hold when $x_1 \leq 0$. By the same token, the independence property will not hold in Bergman's model, in general, since in the latter model the funding of positive positions in risky assets may require borrowing from the cash account B^b .

Proposition 5.7 *Let $x_1 \geq 0$ and Assumptions 3.1 and 3.5 be valid. If a contract (A, C) satisfies Assumption 5.1, then the hedger's price $P_t^h(x_1, A, q)$ is independent of x_1 .*

Proof. From Proposition 5.4, we have $P^h(x_1, A, q) = \bar{Y}^1$ where (\bar{Y}^1, \bar{Z}^1) is the unique solution to the BSDE

$$\begin{cases} d\bar{Y}_t^1 = \bar{Z}_t^{1,*} d\tilde{S}_t^{\text{cld}} + \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) dt + dA_t, \\ \bar{Y}_T^1 = 0. \end{cases}$$

Since $\bar{f}(t, x_1, 0, 0) = 0$ and $A_t - A_0$ is a decreasing process, from the comparison theorem for BSDEs, we obtain $\bar{Y}^1 \geq 0$. Therefore, using the inequalities $x_1 \geq 0$ and $y + q(-y) \geq 0$ for all $y \geq 0$, we get

$$\begin{aligned} \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) &= \sum_{i=1}^d \bar{Z}_t^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{Z}_t^i S_t^i)^+ - x_1 r_t^l B_t^l - r_t^c q(-\bar{Y}_t^1) \\ &\quad + r_t^l (\bar{Y}_t^1 + q(-\bar{Y}_t^1) + x_1 B_t^l + \sum_{i=1}^d (\bar{Z}_t^i S_t^i)^-) \\ &= \sum_{i=1}^d \bar{Z}_t^i \beta_t^i S_t^i - \sum_{i=1}^d r_t^{i,b} (\bar{Z}_t^i S_t^i)^+ - r_t^c q(-\bar{Y}_t^1) \\ &\quad + r_t^l (\bar{Y}_t^1 + q(-\bar{Y}_t^1) + \sum_{i=1}^d (\bar{Z}_t^i S_t^i)^-) \end{aligned}$$

where the last expression is independent of x_1 . Consequently, the price $P_t^h(x_1, A, q) = \bar{Y}_t^1$ is also independent of x_1 . \square

Remark 5.3 Suppose that $x_2 \geq 0$ and a contract (A, C) is such that the process $A - A_0$ is increasing and belongs to $\mathcal{A}(\tilde{\mathbb{P}}^\beta)$. If the collateral C , as seen from the perspective of the hedger, is given by $C_t = q(V_t^c - V_t^0(x_2))$ where the function q satisfies $-y + q(y) \geq 0$ for all $y \geq 0$, then the counterparty's price $P_t^c(x_2, -A, -q)$ is independent of x_2 .

However, if we still work under the assumption of the hedger's collateral, the problem requires more attention, since the counterparty's price depends also on the hedger's initial endowment x_1 . As shown in above proposition, for a contract (A, C) satisfying Assumption 5.1, the process \bar{Y}^1 is independent of x_1 so that, obviously, the price $P_t^c(x_2, -A, -q)$ is independent of x_1 , but it still may depend on x_2 . It is not clear at this moment whether one can find some class of non-trivial contracts (A, C) with the hedger's collateral C given by (3.5) such that $P_t^c(x_2, -A, -q)$ does not depend on x_2 (it may still depend on x_1).

5.4 Positive Homogeneity of the Hedger's Price

We consider once again the hedger's price and we show that it is positively homogeneous with respect to the size of the contract and the non-negative initial endowment. Observe that this property is

no longer true if only the size of the contract, but not the level of the hedger's initial endowment, is inflated (or deflated) through a non-negative scaling factor λ . Of course, this comment does not apply when the price of a contract is known to be independent of the hedger's initial endowment as is the case, for instance, under the assumptions of Proposition 5.7.

Proposition 5.8 *Let $x_1 \geq 0$ and Assumptions 3.1 and 3.5 be valid. For any contract (A, C) such that $A - A_0 \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ and the function q in equation (3.5) is positively homogeneous, meaning that $q(\lambda y) = \lambda q(y)$ for all $\lambda \geq 0$, then the hedger's price is positively homogeneous as well, specifically, for all $\lambda \in \mathbb{R}_+$ and $t \in [0, T]$,*

$$P_t^h(\lambda x_1, \lambda A, q) = \lambda P_t^h(x_1, A, q), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.} \quad (5.13)$$

Proof. It is obvious that (5.13) holds for $\lambda = 0$. Now we suppose that $\lambda > 0$. From Proposition 5.4, we know that $P^h(x_1, A, q) = \bar{Y}^1$ where (\bar{Y}^1, \bar{Z}^1) is the unique solution to the BSDE

$$\begin{cases} d\bar{Y}_t^1 = \bar{Z}_t^{1,*} d\tilde{S}_t^{\text{cld}} + \bar{f}(t, x_1, \bar{Y}_t^1, \bar{Z}_t^1) dt + dA_t, \\ \bar{Y}_T^1 = 0. \end{cases}$$

Similarly, $P^h(\lambda x_1, \lambda A, q) = \tilde{Y}^1$ where $(\tilde{Y}^1, \tilde{Z}^1)$ is the unique solution to the BSDE

$$\begin{cases} d\tilde{Y}_t^1 = \tilde{Z}_t^{1,*} d\tilde{S}_t^{\text{cld}} + \tilde{f}(t, \lambda x_1, \tilde{Y}_t^1, \tilde{Z}_t^1) dt + \lambda dA_t, \\ \tilde{Y}_T^1 = 0. \end{cases}$$

Hence for $\bar{Y} := \lambda \bar{Y}^1$ and $\bar{Z} = \lambda \bar{Z}^1$ we have

$$\begin{cases} d\bar{Y}_t = \bar{Z}_t^* d\tilde{S}_t^{\text{cld}} + \lambda \bar{f}(t, x_1, \lambda^{-1} \bar{Y}_t, \lambda^{-1} \bar{Z}_t) dt + \lambda dA_t, \\ \bar{Y}_T = 0. \end{cases}$$

To complete the proof, it is sufficient to show that for every $\lambda \in \mathbb{R}_+$

$$\lambda \bar{f}(t, x_1, \lambda^{-1} y, \lambda^{-1} z) = \bar{f}(t, \lambda x_1, y, z).$$

This can be checked easily using the property $q(\lambda y) = \lambda q(y)$ for every $\lambda \in \mathbb{R}_+$. \square

If the collateral is given by $C_t = q(V_t^c - V_t^0(x_2))$, then the counterparty's price has the similar positive homogeneity property as in Proposition 5.8. However, if the hedger's collateral is postulated, the study of the homogeneity property of the counterparty's price is slightly more complex, since the counterparty's price depends, in particular, on the hedger's initial endowment x_1 . In that case $C_t = q(V_t^0(x_1) - V_t^h)$, which depends on (x_1, A) , so we shall write $C_t = C_t^{x_1, A}$.

Proposition 5.9 *Let $x_2 \leq 0$ and Assumptions 3.1 and 3.5 be valid. For any contract (A, C) such that $A - A_0 \in \mathcal{A}(\tilde{\mathbb{P}}^\beta)$ and the function q in equation (3.5) is positively homogeneous we have, for all $\lambda \in \mathbb{R}_+$ and $t \in [0, T]$,*

$$P_t^c(\lambda x_2, -\lambda A, C^{\lambda x_1, \lambda A}) = \lambda P_t^c(x_2, -A, C^{x_1, A}), \quad \tilde{\mathbb{P}}^\beta - \text{a.s.}$$

Proof. Similarly as in the proof of Proposition 5.8, it is now sufficient to show that

$$\lambda \bar{g}(t, x_2, \lambda^{-1} y, \lambda^{-1} z) = \bar{g}(t, \lambda x_2, y, z)$$

where the function \bar{g} is given in Proposition 5.4. Since $q(\lambda y) = \lambda q(y)$ and $\tilde{Y}^1 = \lambda \bar{Y}^1$ for $\lambda \geq 0$ (see Proposition 5.8), it is easy to complete the proof. \square

Remark 5.4 Results similar to Propositions 5.8 and 5.9 are also valid when the initial endowments satisfy $x_1 \leq 0$ and $x_2 \geq 0$. Moreover, by combining the results of the last two sections, we can find a class of contracts with prices that are independent of initial endowments and positively homogeneous. Analogous price homogeneity properties can also be established for Bergman's model. The proofs are fairly similar to those for the model with partial netting and thus they are not presented here.

6 Model with Partial Netting and Negotiated Collateral

In the final section, we continue the analysis of the model with partial netting by studying the case where the collateral amount C is negotiated between the counterparties, in the sense that it depends on both the hedger's value $V^h := V(x_1, \varphi, A, C)$ and the counterparty's value $V^c := V(x_2, \tilde{\varphi}, -A, -C)$. As in Section 4, we postulate that the collateral satisfies Assumption 4.1. Recall that in that case we have $P^h(x_1, A, C) = P^h(x_1, x_2, A, C)$ and $P^c(x_2, -A, -C) = P^c(x_1, x_2, -A, -C)$, meaning that the two prices depend on the vector (x_1, x_2) of initial endowments.

6.1 Initial Endowments of Equal Signs

The following result gives fully-coupled pricing BSDEs for both parties under the assumption that their initial endowments are non-negative.

Proposition 6.1 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.3 and 4.1 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\tilde{\mathbb{P}}^l)$ we have $(P^h, P^c)^* = Y$ where (Y, Z) solves the two-dimensional fully-coupled BSDE*

$$\begin{cases} dY_t = Z_t^* d\tilde{S}_t^{l, cld} + g(t, Y_t, Z_t) dt + d\bar{A}_t, \\ Y_T = 0, \end{cases} \quad (6.1)$$

where $g = (g^1, g^2)^*$, $\bar{A} = (A, A)^*$ and for all $y = (y_1, y_2)^* \in \mathbb{R}^2$, $z = (z_1, z_2) \in \mathbb{R}^{d \times 2}$,

$$\begin{aligned} g^1(t, y, z) = & r_t^l (B_t^l)^{-1} z_1^* S_t - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i, b} (z_1^i S_t^i)^+ - x_1 B_t^l r_t^l - r_t^c \hat{q}(-y_1, -y_2) \\ & + r_t^l \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (z_1^i S_t^i)^- \right)^+ \\ & - r_t^b \left(y_1 + \hat{q}(-y_1, -y_2) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (z_1^i S_t^i)^- \right)^- \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} g^2(t, y, z) = & r_t^l (B_t^l)^{-1} z_2^* S_t + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i, b} (-z_2^i S_t^i)^+ + x_2 B_t^l r_t^l - r_t^c \hat{q}(-y_1, -y_2) \\ & - r_t^l \left(-y_2 - \hat{q}(-y_1, -y_2) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z_2^i S_t^i)^- \right)^+ \\ & + r_t^b \left(-y_2 - \hat{q}(-y_1, -y_2) + x_2 B_t^l + (B_t^l)^{-1} \sum_{i=1}^d (-z_2^i S_t^i)^- \right)^-. \end{aligned} \quad (6.3)$$

In the remainder of this section, we work under Assumption 3.4 and we study the valuation and hedging of a European contingent claim (H_T, C) . We note that BSDE (6.1) becomes

$$\begin{cases} dY_t = Z_t \sigma(t, S_t) dW_t + (g(t, Y_t, Z_t) + \sigma(t, S_t) a_t Z_t) dt + d\bar{A}_t, \\ Y_T = 0, \end{cases} \quad (6.4)$$

where the process a is given by (3.15). As in Section 4.3, it suffices to examine the following BSDE

$$\begin{cases} dY_t = Z_t d\tilde{W}_t^l + g(t, Y_t, (\sigma(t, S_t))^{-1} Z_t) dt, \\ Y_T = (-H_T, -H_T)^*. \end{cases}$$

We are now in a position to study the range of fair bilateral prices at time t for a collateralized European claim.

Proposition 6.2 *Let $x_1 \geq 0$, $x_2 \geq 0$ and Assumptions 3.3, 3.4, 4.1 and 4.3 be valid. For any collateralized European claim (H_T, C) where $H_T \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}^l)$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -H_T, -C) \leq P_t^h(x_1, H_T, C), \quad \tilde{\mathbb{P}}^l - \text{a.s.}$$

Proof. Let $\sigma_t^{-1} := (\sigma(t, S_t))^{-1}$. It is sufficient to check that the functions

$$h^1(t, y_1, y_2, z_1, z_2) := -g^1(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

and

$$h^2(t, y^1, y_2, z_1, z_2) := -g^2(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

where g^1 and g^2 are given by (6.2) and (6.3) with $d = 1$, satisfy Assumption 4.2 and condition (4.13). It is easy to check that Assumption 4.2 holds. We will check that condition (4.13) is satisfied as well. We set

$$\delta_1 := y_1^+ + y_2 + \widehat{q}(-y_1^+ - y_2, -y_2) + x_1 B_t^l + (B_t^l)^{-1} \sigma_t^{-1} ((z_1 + z_2) S_t)^-$$

and

$$\delta_2 := -y_2 - \widehat{q}(-y_1^+ - y_2, -y_2) + x_2 B_t^l + (B_t^l)^{-1} \sigma_t^{-1} (-z_2 S_t)^-.$$

Then

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &= -g^1(t, y_1^+ + y_2, y_2, \sigma_t^{-1}(z_1 + z_2), \sigma_t^{-1} z_2) + g^2(t, y_1^+ + y_2, y_2, \sigma_t^{-1}(z_1 + z_2), \sigma_t^{-1} z_2) \\ &= -r_t^l (B_t^l)^{-1} \sigma_t^{-1} z_1 S_t + (B_t^l)^{-1} r_t^{1,b} (\sigma_t^{-1}(z_1 + z_2) S_t)^+ + (B_t^l)^{-1} r_t^{1,b} (-\sigma_t^{-1} z_2 S_t)^+ \\ &\quad + (x_1 + x_2) B_t^l r_t^l - r_t^l (\delta_1^+ + \delta_2^+) + r_t^b (\delta_1^- + \delta_2^-). \end{aligned}$$

Since $r_t^l \leq r_t^b$, we have

$$\begin{aligned} & r_t^l (\delta_1^+ + \delta_2^+) - r_t^b (\delta_1^- + \delta_2^-) \leq r_t^l (\delta_1 + \delta_2) \\ &= r_t^l y_1^+ + (x_1 + x_2) B_t^l r_t^l + r_t^l (B_t^l)^{-1} ((\sigma_t^{-1}(z_1 + z_2) S_t)^- + (-\sigma_t^{-1} z_2 S_t)^-). \end{aligned}$$

Thus, using $r_t^{1,b} \geq r_t^l$, we obtain

$$\begin{aligned} & h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) \\ &\geq -r_t^l y_1^+ - r_t^l (B_t^l)^{-1} \sigma_t^{-1} z_1 S_t + (B_t^l)^{-1} r_t^{1,b} (\sigma_t^{-1}(z_1 + z_2) S_t)^+ + (B_t^l)^{-1} r_t^{1,b} (-\sigma_t^{-1} z_2 S_t)^+ \\ &\quad - r_t^l (B_t^l)^{-1} ((\sigma_t^{-1}(z_1 + z_2) S_t)^- + (-\sigma_t^{-1} z_2 S_t)^-) \\ &= -r_t^l y_1^+ + (B_t^l)^{-1} (r_t^{1,b} - r_t^l) (\sigma_t^{-1}(z_1 + z_2) S_t)^+ + (B_t^l)^{-1} (r_t^{1,b} - r_t^l) (-\sigma_t^{-1} z_2 S_t)^+ \geq -r_t^l y_1^+. \end{aligned}$$

Using similar arguments as in the proof of Proposition 4.4, we conclude that (4.13) holds. \square

6.2 Initial Endowments of Opposite Signs

We conclude the paper by studying the case of initial endowments of opposite signs.

Proposition 6.3 *Let $x_1 \geq 0$, $x_2 \leq 0$ and Assumptions 3.5 and 4.1 be valid. For any contract (A, C) such that $A \in \mathcal{A}(\widetilde{\mathbb{P}}^\beta)$ we have $(P^h, P^c)^* = \widehat{Y}$ where $(\widehat{Y}, \widehat{Z})$ solves the two-dimensional fully-coupled BSDE*

$$\begin{cases} d\widehat{Y}_t = \widehat{Z}_t^* d\widetilde{S}_t^{cld} + \widehat{g}(t, \widehat{Y}_t, \widehat{Z}_t) dt + d\overline{A}_t, \\ \widehat{Y}_T = 0, \end{cases} \quad (6.5)$$

where $\widehat{g} = (\widehat{g}^1, \widehat{g}^2)^*$, $\overline{A} = (A, A)^*$ and for all $y = (y_1, y_2)^* \in \mathbb{R}^2$, $z = (z_1, z_2) \in \mathbb{R}^{d \times 2}$,

$$\begin{aligned} \widehat{g}^1(t, y, z) = & \sum_{i=1}^d z_1^i \beta_t^i S_t^i - x_1 B_t^l r_t^l - r_t^c \widehat{q}(-y_1, -y_2) \\ & + r_t^l \left(y_1 + \widehat{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \right)^+ \\ & - r_t^b \left(y_1 + \widehat{q}(-y_1, -y_2) + x_1 B_t^l - z_1^* S_t \right)^- \end{aligned} \quad (6.6)$$

and

$$\begin{aligned}\widehat{g}^2(t, y, z) = & \sum_{i=1}^d z_1^i \beta_t^i S_t^i + (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (-z_2^i S_t^i)^+ + x_2 B_t^b r_t^b - r_t^c \widehat{q}(-y_1, -y_2) \\ & - r_t^l \left(-y_2 - \widehat{q}(-y_1, -y_2) + x_2 B_t^b + (B_t^l)^{-1} \sum_{i=1}^d (z_2^i S_t^i)^- \right)^+ \\ & + r_t^b \left(-y_2 - \widehat{q}(-y_1, -y_2) + x_2 B_t^b + (B_t^l)^{-1} \sum_{i=1}^d (z_2^i S_t^i)^- \right)^-.\end{aligned}\quad (6.7)$$

Let Assumptions 3.4 and 3.5 be satisfied. Then

$$d\widetilde{S}_t^{\text{old}} = (\mu(t, S_t) + \kappa(t, S_t) - \beta_t S_t) dt + \sigma(t, S_t) dW_t$$

and BSDE (6.5) becomes

$$\begin{cases} dY_t = Z_t \sigma(t, S_t) dW_t + (\widehat{g}(t, Y_t, Z_t) + \sigma(t, S_t) b_t Z_t) dt + d\overline{A}_t, \\ Y_T = 0, \end{cases}$$

where the process b is given by (4.18). As in Section 4.3, it suffices to examine the following BSDE

$$\begin{cases} dY_t = Z_t d\widetilde{W}_t^\beta + \widehat{g}(t, Y_t, (\sigma(t, S_t))^{-1} Z_t) dt, \\ Y_T = (-H_T, -H_T)^*, \end{cases}$$

where \widetilde{W}^β is a Brownian motion under an equivalent probability measure $\widetilde{\mathbb{P}}^\beta$.

Proposition 6.4 *Let $x_1 \geq 0, x_2 \leq 0$ be such that $x_1 x_2 = 0$. If Assumptions 3.4, 3.5, 4.1 and 4.4 are met, then for any collateralized European claim (H_T, C) such that $H_T \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}}^\beta)$ we have, for every $t \in [0, T]$,*

$$P_t^c(x_2, -H_T, -C) \leq P_t^h(x_1, H_T, C), \quad \widetilde{\mathbb{P}}^\beta - \text{a.s.}$$

Proof. We write, as usual, $\sigma_t^{-1} := (\sigma(t, S_t))^{-1}$. It is sufficient to check that the functions

$$h^1(t, y_1, y_2, z_1, z_2) := -\widehat{g}^1(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

and

$$h^2(t, y^1, y_2, z_1, z_2) := -\widehat{g}^2(t, y_1, y_2, \sigma_t^{-1} z_1, \sigma_t^{-1} z_2)$$

where \widehat{g}^1 and \widehat{g}^2 are given by (4.7) and (4.8), respectively, satisfy Assumption 4.2 and condition (4.13). This is similar to the proof of Proposition 4.7, using $x_1 x_2 = 0$ and the same computations as in the proof of Proposition 6.2. The details are left to the reader. \square

Acknowledgement.

The research of Tianyang Nie and Marek Rutkowski was supported under Australian Research Council's Discovery Projects funding scheme (DP120100895).

References

- [1] Bergman, Y. Z.: Option pricing with differential interest rates. *Review of Financial Studies* 8 (1995), 475–500.
- [2] Bielecki, T. R., Rutkowski, M.: Valuation and hedging of contracts with funding costs and collateralization. Working paper, 2014.
- [3] Brigo, D., Capponi, A., Pallavicini, A., Papatheodorou, V.: Collateral margining in arbitrage-free counterparty valuation adjustment including re-hypothecation and netting. Working paper, 2011.

- [4] Buckdahn, R., Quincampoix, M., Rascanu, A.: Viability property for a backward stochastic differential equations and applications to partial differential equations. *Probab. Theory Related Fields* 116 (2000), 485–504.
- [5] Burgard, C., Kjaer, M.: PDE representations of options with bilateral counterparty risk and funding costs. Working paper, November, 2009.
- [6] Burgard, C., Kjaer, M.: Partial differential equations representations of derivatives with counterparty risk and funding costs. *Journal of Credit Risk* 7 (2011), 1–19.
- [7] Crépey, S.: Bilateral counterparty risk under funding constraints – Part I: Pricing. *Mathematical Finance* (published online on 12 December 2012).
- [8] Crépey, S.: Bilateral counterparty risk under funding constraints – Part II: CVA. *Mathematical Finance* (published online on 12 December 2012).
- [9] El Karoui, N., Peng, S., Quenez, M. C.: Backward stochastic differential equations in finance. *Mathematical Finance* 7 (1997), 1–71.
- [10] Hu, Y., Peng, S.: On the comparison theorem for multidimensional BSDEs. *C.R. Acad. Sci. Paris. Ser. I.* 343 (2006) 135–140.
- [11] Mercurio, F.: Bergman, Piterbarg and beyond: Pricing derivatives under collateralization and differential rates. Working paper, 2013.
- [12] Nie, T., Rutkowski, M.: Fair and profitable bilateral prices under funding costs and collateralization. Working paper, University of Sydney, 2014.
- [13] Nie, T., Rutkowski, M.: BSDEs driven by a multi-dimensional martingale and their applications to market models with funding costs. Working paper, University of Sydney, 2014.
- [14] Nie, T., Rutkowski, M.: Fair bilateral prices in Bergman’s model. Working paper, University of Sydney, 2014.
- [15] Pallavicini, A., Perini, D., Brigo, D.: Funding, collateral and hedging: uncovering the mechanics and the subtleties of funding valuation adjustments. Working paper, 2012.
- [16] Piterbarg, V.: Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, February (2010), 97–102.